Midterm 1 Solutions
MA 265
Instructor: Javier Zuniga NAME: ______________________

Here are the solutions. Remember that usually there is more than one way to get to the answer.

1. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

   Compute

   (a) $A \cdot A - 3A^T = \begin{bmatrix} -4 & 5 & 2 \\ -7 & 1 & -10 \\ -3 & -6 & 7 \end{bmatrix}$

   (b) $\det\left[(\frac{1}{2}A^T) \cdot (-A^{-1})\right] = \left(\frac{1}{2}\right)^3 \det(A^T) \cdot (-1)^3 \det(A^{-1}) = -\frac{1}{8}\det(A) \cdot \det(A^{-1}) = -\frac{1}{8}\det(A \cdot A^{-1}) = -\frac{1}{8}\det(I_3) = -\frac{1}{8}$

2. (10 points) If $A\vec{x} = \vec{b}$ determine whether the system is consistent, has a unique solution or an infinite number of solutions.

   (a) $A = \begin{bmatrix} 1 & -7 & 0 \\ 1 & -1 & 1 \\ -2 & 8 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   The REF of the system is

$$\begin{bmatrix} 1 & -7 & 0 & 1 \\ 0 & 1 & 1/6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   and thus the system is inconsistent.

   (b) $A = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}$ and $\vec{b} = \vec{0}$

   Since $\det A = 2 \neq 0$ the system has a unique solution.

3. (10 points) Show that if A is a symmetric matrix then

   (a) $AA \cdots A$ (k times) is also symmetric It is enought to check that A^k is symmetric, that is

$$(A^k)^T = (A^T)^k = A^k$$

   since $A^T = A$.

   (b) A^{-1} is also symmetric provided A is non-singular. Again, since $A^T = A$ we have

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

4. (10 points) Find the adjoint of

$$\begin{bmatrix} -2 & 0 & 1 \\ 3 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

Recall that the adjoint is the traspose of the matrix of cofactors. Thus

$$adj\,A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 5 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

5. (20 points) True or False?

(a) The set of continuous functions in one variable with usual sum and product such that $f(1) = 0$ is a vector space. **T**

(b) The set of differentiable functions in one variable with usual sum and product such that $f'(0) = 1$ is a vector space. **F**

(c) The set of n by n matrices with usual sum and product whose sum of entries on the last row adds up to zero is a subspace of $M_{n,n}$. **T**

(d) The set of polynomials with only even powers of t is a subspace of the vector space of polynomials. **T**

(e) The set of rational numbers is a subspace of $\mathbb{R}^1$. **F**

(f) The subset of $\mathbb{R}^{100}$ whose sum of odd entries equal to zero is a subspace of $\mathbb{R}^{100}$. **T**

(g) The line passing through the points $(1, 1)$ and $(-1, -1)$ is a subspace of $\mathbb{R}^2$. **T**

(h) The set of invertible matrices is a subspace of $M_{n,n}$. **F**

(i) The plane in $\mathbb{R}^3$ with equation $x + y + z = 1$ is a subspace of $\mathbb{R}^3$. **F**

(j) The set of points (x, y) in the parabola $y = x^2$ is a subspace. **F**

6. (20 points) Let $S = \{1+t, t+t^2, -1+t^2, 1+2t+t^2, 3-t-2t^2\}$ Is span $S = P_2$? Find a subset R of these polynomials that is linealry independent and such that span $R =$ span S.

To check for the span we construct the following system

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 3 & a \\ 1 & 1 & 0 & 2 & -1 & b \\ 0 & 1 & 1 & 1 & -2 & c \end{bmatrix}$$

whose REF is

$$\left[\begin{array}{cccccc} 1 & 0 & -1 & 1 & 3 & a \\ 0 & 1 & 1 & 1 & -2 & c \\ 0 & 0 & 0 & 0 & 1 & \frac{a+c-b}{2} \end{array}\right]$$

Since the system is consistent the set spans P_2.

To find redundant vectors it is enought to test for linear independence and get rid of the vectors corresponding with free variables. The RREF of the system obtained tells us that the second and third columns correspond with free variables. Therefore $R = \{1+t, t+t^2, 3-t-2t^2\}$

7. (20 points) Show that the set of 2 by 2 symmetric matrices is a subspace of $M_{2,2}$. Find a finite set of generators of this subspace.

   Let $W = \left\{ \left[\begin{array}{cc} a & b \\ b & c \end{array}\right]; \quad a, b, c \in \mathbb{R} \right\}$ be such set. We first check the three properties of a subspace.

   (a) The zero matrix $\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$ is clearly symmetric and thus belong to W.

   (b) $\left[\begin{array}{cc} a & b \\ b & c \end{array}\right] + \left[\begin{array}{cc} a' & b' \\ b' & c' \end{array}\right] = \left[\begin{array}{cc} a+a' & b+b' \\ b+b' & c+c' \end{array}\right]$ is clearly symmetric.

   (c) $k \left[\begin{array}{cc} a & b \\ b & c \end{array}\right] = \left[\begin{array}{cc} ka & kb \\ kb & kc \end{array}\right]$ is also symmetric.

   To find a set of generators is enough to realize that

$$\left[\begin{array}{cc} a & b \\ b & c \end{array}\right] = a \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + b \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] + c \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]$$

   which means that the set $\left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] \right\}$ is a set of generators.