MA265 Midterm 2 Solutions, Spring 08
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1. (10 points) Let W be a subspace of $M_{2,2}$ spanned by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & k \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 \\ k & 2 \end{bmatrix}$. Determine the values of k so that $W^\perp$ has dimension zero.

   Since $\dim M_{2,2} = 4$ it is enough to require the determinant to be different from zero. But then

$$\det \begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & k & 1 & -1 \\ 0 & 1 & 0 & k \\ 1 & 0 & -1 & 2 \end{vmatrix} = \det \begin{vmatrix} k & 1 & -1 \\ 1 & 0 & k \\ 0 & -1 & 2 \end{vmatrix} - \det \begin{vmatrix} 1 & 1 & 0 \\ k & 1 & -1 \\ 1 & 0 & k \end{vmatrix} = 2k^2 - k = k(2k-1)$$

   implies that k must be different from zero or $1/2$.

2. (10 points) Show that if V is any vector space and $(,)$ is any inner product then the following properties hold. (HINT 1: Showing this for particular examples won't give you any credit, HINT 2: $\|\vec{u}\|^2 = (\vec{u}, \vec{u})$)

   (a) If $(\vec{u}, \vec{v}) = 0$ then $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

$$\begin{aligned} \|\vec{u} + vecv\|^2 &= (\vec{u} + \vec{v}, \vec{u} + \vec{c}) = (\vec{u}, \vec{u}) + 2(\vec{u}, \vec{v}) + (\vec{v}, \vec{v}) \\ &= (\vec{u}, \vec{u}) + (\vec{v}, \vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 \end{aligned}$$

   (b) If $(\vec{u} - \vec{v}, \vec{v} - \vec{u}) = 0$ then $\vec{u} = \vec{v}$.
   Since

$$(\vec{u} - \vec{v}, \vec{v} - \vec{u}) = -\|\vec{u} - \vec{v}\|^2$$

   then $(\vec{u} - \vec{v}, \vec{v} - \vec{u}) = 0$ implies $-\|\vec{u} - \vec{v}\|^2 = 0$. But that can only be true if the vector inside is zero by the axioms of the inner product. Hence $\vec{u} - \vec{v} = \vec{0}$ or $\vec{u} = \vec{v}$.

3. (10 points) For which values of k is the rank of the following matrix equal to two?

$$\begin{bmatrix} -1 & 2-k & 3 & k-4 \\ 0 & 2 & 4 & 2k-6 \\ 1 & k-1 & -1 & 1 \end{bmatrix}$$

   It is possible to row reduce the matrix to the following form

$$\begin{bmatrix} 1 & k-1 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   This means that no matter what k is we always have two pivot columns and hence rank two.

4. (20 points) True or False?

(a) A linear system of 6 equations with 4 variables can have a unique solution.
True, since we have more equations than variables, this is possible.

(b) A homogenous system of ten equations in five variables is always consistent.
True, because the system is homogeneous the trivial solution is always a solution.

(c) Five vectors can span $\mathbb{R}^6$.
False, we need at least six vectors.

(d) $L(x, y, z) = \begin{bmatrix} x^2 & x - y \\ -z & 0 \end{bmatrix}_{2,2}$ is a linear transformation.
False, the quadratic term on the first entry will allow us to get counter examples.

(e) $L(x, y, z) = (z)t^2 - (x + y)t + (2z)$ is a linear transformation.
True (try to prove it!).

(f) $L(x, y) = (2y, \frac{1}{2}x)$ is an isometry.
False.

(g) $L(x, y) = (\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}})$ is an isometry.
True.

(h) If nullity of $A_{5,10}$ is 7 then rank $A = 3$.
True, because rank + nullity = 10.

(i) It is possible for $A_{3,7}$ to have rank equal to four.
False, the rank of A is at most three.

(j) If nullity of $A_{8,4}$ is two then the dimension of $(ColA)^\perp$ is six.
True, since nullity of A is 2 this implies that the rank is also 2. But then nullity of A^T is six and since $(ColA)^\perp = Null\, A^T$ the claim follows.

5. (20 points) Let

$$L(x, y) = \left(\frac{5x + y}{2}, \frac{x + 5y}{2} \right)$$

Set A to be the matrix associated to this linear transformation. Find a diagonal matrix D and an orthogonal matrix P such that $P^{-1}AP = D$.

The associated matrix is

$$\begin{bmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{bmatrix}$$

and thus the characteristic polynomial is $\lambda^2 - 5\lambda + 6$. The eigenvalues are 2 and 3. The associated eigenvectors are the bases for the nullspaces of the matrices

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \quad \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

. Therefore the associated eigenvectors are (1,-1) and (1,1). They are obviously orthogonal (this was expected since the matrix A is symmetric). Thus

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

6. (15 points) Let $W = \text{Span}\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix}\right\}$.

(a) (5 points) Find a basis for $W^\perp$.

This is the same as solving $x + 2y + z = 0$ in $\mathbb{R}^4$. Thus y, z and w are free variables and

$$\begin{bmatrix}x\\y\\z\\w\end{bmatrix} = \begin{bmatrix}-2y-z\\y\\z\\w\end{bmatrix} = y\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + z\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + w\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Thus a basis for $W^\perp$ is

$$\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

(b) (10 points) Use the Gram-Schmidt process to find an orthonormal basis for $W^\perp$.

The first two vectors of the previous basis are not orthogonal, but the last two are. It would then be more convinient to use Gram-Schimdt on the following basis

$$\left\{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}\right\}$$

Since the first two vectors are orthogonal

$$\vec{v}_1 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

The last vector can be computed by doing

$$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \begin{bmatrix}-2\\1\\0\\0\end{bmatrix} - \frac{0}{1}\vec{v}_1 - \frac{2}{2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\-1\\0\end{bmatrix}$$

Since we are require to give an orthonormal basis we normalize to obtain

$$\left\{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\-1\\0\end{bmatrix}\right\}$$

7. (15 points) Let W be as in the previous problem.

(a) (10 points) Find the projection of $\begin{bmatrix} -1 \\ -2 \\ -1 \\ 1 \end{bmatrix}$ onto $W^{\perp}$.

Let $\vec{u} = \begin{bmatrix} -1 \\ -2 \\ -1 \\ 1 \end{bmatrix}$ Since the basis on the previous problem was orthonormal the projection can be computed by

$$proj_W \vec{u} = (\vec{u} \cdot \vec{v}_1)\vec{v}_1 + (\vec{u} \cdot \vec{v}_2)\vec{v}_2 + (\vec{u} \cdot \vec{v}_3)\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) (5 points) Find the distance from $\begin{bmatrix} -1 \\ -2 \\ -1 \\ 1 \end{bmatrix}$ to $W^{\perp}$.

The requirec distance is

$$\left\| \begin{bmatrix} -1 \\ -2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ -2 \\ -1 \\ 0 \end{bmatrix} \right\| = \sqrt{6}$$