Practice Midterm 1 with Solutions
MA 265
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No calculators or notes allowed. There are a total of 100 points. Read the exam carefully. Show your work for full credit.

1. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

Compute

(a) $2A - A^T \cdot A$

$$\begin{bmatrix} 0 & 2 & 1 \\ -4 & -13 & 5 \\ 1 & 9 & -4 \end{bmatrix}$$

(b) $\det\left[(3A^T)\cdot(-A)\cdot(A^2)\right]$
Since $\det A = -5$ then

$$\begin{aligned}\det\left[(3A^T)\cdot(-A)\cdot(A^2)\right] &= \det[3A^T]\det[-A]\det[A^2] \\ &= 3^3\det[A^T](-1)^3\det[A]\det[A]^2 \\ &= -3^3(-5)^4 = -3^3\cdot 5^4\end{aligned}$$

2. (10 points) If $A\vec{x} = \vec{b}$ determine whether the system is inconsistent, has a unique solution or an infinite number of solutions.

(a) $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & -4 & 1 \\ -2 & 8 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
Put the augmented matrix in REF to check that the system is inconsistent.

(b) $A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ and $\vec{b} = \vec{0}$
We can put the augmented matrix in REF to notice that the system has a unique solution. Another way to prove this is to realize that since A is square and the system is homogeneous it is enough to compute the determinant. Since $\det A = 1$ the system has a unique solution.

3. (10 points) Show that if A is a square matrix then

(a) $A + A^T$ is symmetric

$$(A + A^T)^T = (A)^T + (A^T)^T = A^T + A = (A + A^T)$$

(b) $A - A^T$ is skew-symmetric

$$(A - A^T)^T = (A)^T - (A^T)^T = A^T - A = -(A - A^T)$$

(c) if A is skew-symmetric then A^T is also skew-symmetric.

$$-(A^T)^T = -A = -(-A^T) = (A^T)$$

4. (10 points) Find the adjoint of

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

5. (20 points) True or False?

(a) The set of twice differentiable real functions is a vector space.
TRUE (differentiation is linear, no matter what the order of the derivative is)

(b) The set of continuous functions is a subspace of the vector space of differentiable functions.
FALSE (is the other way around, the set of differentiable functions is a subspace of the space of continuous functions)

(c) The set of m by n matrices whose sum of entries on the first column adds up to zero is a subspace of $M_{m,n}$.
TRUE (check this!)

(d) The set of m by n matrices whose sum of entries on the first row is equals to 1 is a subspace of $M_{m,n}$.
FALSE (the zero matrix doesn't satisfy this property)

(e) The set of positive real numbers with sum defined as $x \oplus y = x \cdot y$ and $c \odot x = x^c$ is a subspace of $\mathbb{R}^1$.
FALSE (the operations are not the same!)

(f) The subset of $\mathbb{R}^n$ with last entry equal to zero is a subspace.
TRUE (check this!)

(g) The line passing through the points $(1, 1)$ and $(1, 0)$ is a subspace of $\mathbb{R}^2$.
FALSE (it doesn't include the origin)

(h) The set of matrices with zero determinant is a subspace of $M_{2,2}$.
FALSE ($\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ both have zero determinant but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has determinant equal to 1. Therefore this set is not closed under addition)

(i) The plane in $\mathbb{R}^3$ with equation $2x + 3y - z = 0$ is a subspace of $\mathbb{R}^3$.
TRUE (this plane includes the origin!)

(j) The set of solutions to an non-homogeneous system is a subspace.
FALSE (the zero vector can not be a solution, so it doesn't belong to this set!)

6. (10 points) Do the polynomials $t^3 + 2t + 1$, $t^2 - t + 2$, $t^3 + 2$, $-t^3 + t^2 - 5t + 2$, $t + 1$ span P_3?

Yes, to show this write down the matrix whose columns are the coefficients of the polynomials. Putting this matrix in REF gives four pivot columns and this ensures that these five vectors span P_3.

7. (20 points) Show that the set of 2 by 2 matrices whose sum of all entries is equal to zero is a subspace of $M_{2,2}$. Find a set of matrices that spans this subspace and that is linearly independent. Can you find another set of spanning matrices that is a subset of the given set of matrices?

$$\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 4 \\ -7 & -3 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

To show that it is a subspace we check three things. 1) It is not empty since the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has as sum of entries: $0 + 0 + 0 + 0 = 0$. 2) It is closed under addition because if $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ are two such matrices then the sum $\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$ satisfies

$(a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{21} + b_{21}) + (a_{22} + b_{22}) = (a_{11} + a_{12} + a_{21} + a_{22}) + (b_{11} + b_{12} + b_{21} + b_{22}) = 0 +$

3) Finally if c is a constant and $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is such matrix then the re-scaled matrix $\begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$ satisfies

$$ca_{11} + ca_{12} + ca_{21} + ca_{22} = c(a_{11} + a_{12} + a_{21} + a_{22}) = c0 = 0$$

Given a 2 by 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a + b + c + d = 0$ solving for d gives $d = -a - b - c$. Thus

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a - b - c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

and the set of spanning matrices is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

If we try to find another set of spanning matrices that is a subset of the given matrices we set a system

$$c_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 6 & 4 \\ -7 & -3 \end{bmatrix} + c_5 \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $d = -a - b - c$

The associated augmentmented matrix is

$$\begin{bmatrix} 1 & 2 & 0 & 6 & -1 & a \\ -1 & 2 & 1 & 4 & 0 & b \\ -1 & -2 & -1 & -7 & 1 & c \\ 1 & -2 & 0 & -3 & 0 & -a-b-c \end{bmatrix}$$

which is row equivalent to

$$\begin{bmatrix} 1 & 2 & 0 & 6 & -1 & a \\ 0 & 4 & 1 & 10 & -1 & a+b \\ 0 & 0 & 1 & 1 & 0 & -a-c \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we can take the first three vectors on the given set since they correspond with pivot columns and the system is consistent.

8. (10 points) Find a set of vectors spanning the solution space of $A\vec{x} = \vec{0}$ where

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

The given matrix is row equivalent to

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the solution is $x_1 = -r$, $x_2 = -r$, $x_3 = r$, $x_4 = 0$. The solution is then spanned by the vector

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$