Practice Midterm 2 Solutions
MA 265
Instructor: Javier Zuniga

1. (10 points) For which values of α is the set

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \alpha - 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3\alpha \\ 4 \\ 2 \\ \alpha \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}$$

linearly independent?

Four vectors in $\mathbb{R}^4$ are linearly independent if the determinant of the matrix whose columns correspond with those vectors is different from zero. Since the determinant (which can be computed by expanding along the first column) is

$$\det \begin{vmatrix} 2 & 1 & 3\alpha & 4 \\ 0 & \alpha - 1 & 4 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & \alpha & 4 \end{vmatrix} = 2(\alpha - 1)(8 - \alpha)$$

it is enough to ask for α to be different from 1 or 8.

2. (10 points) Find the dimension of the following subspaces

(a) All 2 by 2 matrices of the form $\begin{bmatrix} a+b & c+d \\ d+e & a+b \end{bmatrix}$.

Since

$$\begin{bmatrix} a+b & c+d \\ d+e & a+b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

we can eliminate the second and fourth vectors (they are clearly linear combinations of the others). The resulting set of three vectors is linearly independent (why?). Thus the dimension is three. You can also write down the associated whose columns are these matrices and find the REF to get the same answer.

(b) All polynomials of the form $at^2 + (a+b)t + (a+b+c+d)$.

In this case $at^2 + (a+b)t + (a+b+c+d) = a(t^2+t+1) + b(t+1) + c(1) + d(1)$. Thus the last vector is redundant (this is already in REF) and thus the dimensions is again three.

(c) All vectors of the forms $\begin{bmatrix} a+c \\ a-b \\ b+c \\ -a+b \end{bmatrix}$

Here

$$\begin{bmatrix} a+c \\ a-b \\ b+c \\ -a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and since the third vector is the sum of the first two we can eliminate it. The first two vectors are clearly linearly indendent and thus they are a basis. This means that the dimension is two in this case.

3. (10 points) Show that if V is any vector space and $(,)$ is an inner product then
   (a) $(\vec{u}, \vec{0}) = 0$, for any vector $\vec{u}$ in V.
   Since $\vec{0} = \vec{0} + \vec{0}$ we have
   $$(\vec{u}, \vec{0}) = (\vec{u}, \vec{0} + \vec{0}) = (\vec{u}, \vec{0}) + (\vec{u}, \vec{0})$$
   and thus cancelling on both sides gives $(\vec{u}, \vec{0}) = 0$.
   (b) $||c \cdot \vec{u}|| = |c| \cdot ||\vec{u}||$ for any constant $c \in \mathbb{R}$.
   $$||c \cdot \vec{u}|| = \sqrt{(c\vec{u}, c\vec{u})} = \sqrt{c^2(\vec{u}, \vec{u})} = |c|\sqrt{(\vec{u}, \vec{u})} = |c| \cdot ||\vec{u}||$$
   (c) $||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$ for any couple of vectors.
   Recall that the Cauchy-Schwarz iquality states that $|(\vec{u}, \vec{v})| \leq ||\vec{u}|| \cdot ||\vec{v}||$. Therefore,
   $$\begin{aligned} ||\vec{u} + \vec{v}||^2 &= (\vec{u} + \vec{v}, \vec{u} + \vec{v}) = (\vec{u}, \vec{u}) + 2(\vec{u}, \vec{v}) + (\vec{v}, \vec{v}) = ||\vec{u}||^2 + 2(\vec{u}, \vec{v}) + ||\vec{v}|| \\ &\leq ||\vec{u}||^2 + 2||\vec{u}|| \cdot ||\vec{v}|| + ||\vec{v}||^2 = (||\vec{u}|| + ||\vec{v}||)^2 \end{aligned}$$
   And taking the square root of both sides gives the required inequality.

4. (10 points) Find the rank of the following matrices
   (a) $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 4 & -2 & 0 \end{bmatrix}$
   The REF of this matrix is the same as the one in part (b) with the last column removed. This doesn't affect the number of pivot columns so the rank is the same, that is rank = 3.
   (b) $\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 2 & 0 & 3 & 1 & 2 \\ 0 & 4 & -2 & 0 & -1 \end{bmatrix}$
   The REF of the given matrix is
   $$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{4} \end{bmatrix}$$
   and thus has rank = 3.
   (c) Is the system
   $$\begin{aligned} a - b + c - d &= 1 \\ 2a + 3c + d &= 2 \\ 4b - 2c &= -1 \end{aligned}$$
   consistent?
   The coefficient matrix is equal to the matrix in part (a). The augmented matrix is equal to the matrix in part (b). Since they have the same rank the system is consistent.

5. (20 points) True or False?

(a) A linear system of three equations with four variables can have a unique solution.

False. There is always at least one free variable (check this!) and hence an infinite number of solutions.

(b) A linear system of three equations in two variables is always consistent.

False. Any system of any size can be inconsistent.

(c) Five vectors in $\mathbb{R}^4$ can not be linearly indenpendent.

True. Otherwise $4 = \dim \mathbb{R}^4 \geq 5$ which is imposible.

(d) $L(x, y, z) = \begin{bmatrix} x+y & x-y \\ 0 & 2z \end{bmatrix}_{2,2}$ is a linear transformation.

True. Check that it preserves sum and re-scaling.

(e) $L(x, y, z) = t^2 - (x+y)t + (2z)$ is a linear transformation.

False. Is enough to realize that $L(\vec{0}) = t^2 \neq 0$.

(f) $L(x, y) = (-y, -x)$ is an isometry.

True. We have that

$$L(x_1, y_1) \cdot L(x_2, y_2) = (-y_1, -x_1) \cdot (-y_2, -x_2) = y_1y_2 + x_1x_2 = x_1x_2 + y_1y_2 = (x_1, y_1) \cdot (x_2, y_2)$$

(g) $L(x, y) = (x-y, x+y)$ is an isometry.

False. In this case it can be shown that

$$L(x_1, y_1) \cdot L(x_2, y_2) = 2(x_1, y_1) \cdot (x_2, y_2) \neq (x_1, y_1) \cdot (x_2, y_2)$$

(h) If rank $A_{3,4}$=3 then any basis of for Null A has only one vector.

True. Rank+Nullity =4 so nullity = 4-3= 1.

(i) If nullity of $A_{6,4}$ is 3 then rank $A = 3$.

False. Rank+Nullity =4 so Rank=4-3=1.

(j) If a plane in $\mathbb{R}^4$ is the span of two l.i. vectors, then the orthogonal complement of a plane in $\mathbb{R}^4$ is also a plane.

True. Here a plane is defined as a vector space of dimension two. If P is such a plane then dim P + dim $P^\perp = 4$. But then dim $P^\perp = 2$ and $P^\perp$ is also a plane.

6. (10 points) Let W be a subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ k \\ 2 \end{bmatrix}, \begin{bmatrix} k+8 \\ 5 \\ 4 \end{bmatrix}$. Determine the values of k so that $W^\perp$ has dimension zero.

The dimension of $W^\perp$ is zero when the dimension of W is exactly three (this is because of the formula $\dim W + \dim W^\perp = \dim \mathbb{R}^3$). For W to have dimension three we need the three given vectors to be linearly independent. But remember that we have a nice condition for three vectors to be linearly indenpendent in $\mathbb{R}^3$: it is enough to check that the determinant whose columns are the given vectors is different from zero. Thus

$$\det \begin{vmatrix} 3 & 3 & k+8 \\ 2 & k & 5 \\ 1 & 2 & 4 \end{vmatrix} = -k^2 + 8k - 7 = -(k-1)(k-7) \neq 0$$

which it's try only when $k \neq 1$ and $k \neq 7$.

7. (15 points) Compute the distance from the plane $x + y + z = 1$ to the point (-1,-1,2).

We need to compute the norm of the difference between the given point and its projection to the given plane. First we need to find an orthogonal basis for the plane. Since $x = -y - z + 1$, $y = r$ and $z = s$ are free variables we have that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Since the plane is shifted by the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ the distance we are looking for is the same as the distance from the plane Span S where $S = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ to the point $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$. Now we need to compute the projection but S is not orthogonal, so we apply Gram-Schimdt to S. Thus take $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

But the last vector has fractions in it. So it is better to take $\vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

Now that we have an orthogonal set we can compute the projection

$$\operatorname{proj}_{\{\vec{v}_1, \vec{v}_2\}} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \frac{2-1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \frac{2+1+2}{1+1+4} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4/3 \\ -1/3 \\ 5/3 \end{bmatrix}$$

Finally, the distance is computed by

$$\left\|\begin{bmatrix}-4/3\\-1/3\\5/3\end{bmatrix}-\begin{bmatrix}-2\\-1\\2\end{bmatrix}\right\|=\left\|\begin{bmatrix}2/3\\2/3\\-1/3\end{bmatrix}\right\|=\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}}=1$$

8. (15 points) Let $L(x, y, z) = (-2z, -2y, -2x + 3z)$. Set A to be the matrix associated to this linear transformation. Find a diagonal matrix D and an orthogonal matrix P such that $P^{-1}AP = D$.

First we need to find the matrix A associated to the linear transformation. Since $L(1, 0, 0) = (0, 0, -2)$, $L(0, 1, 0) = (0, -2, 0)$ and $L(0, 0, 1) = (-2, 0, 3)$ we conclude that

$$A=\begin{bmatrix}0&0&-2\\0&-2&0\\-2&0&3\end{bmatrix}$$

The characteristic polynomial is

$$\begin{aligned}\det\begin{vmatrix}-\lambda&0&-2\\0&-2-\lambda&0\\-2&0&3-\lambda\end{vmatrix}&=-\lambda(-\lambda-2)(-\lambda+3)-4(-\lambda-2)\\&=(-\lambda^2+3\lambda)(\lambda+2)+(4)(\lambda+2)\\&=-(\lambda^2-3\lambda-4)=-(\lambda+2)(\lambda-4)(\lambda+3)\end{aligned}$$

The eigenvalues are then $\lambda_1 = -2$, $\lambda_2 = 4$ and $\lambda_3 = -3$. Since the matrix A is symmetric we know the corresponding eigenvectors will be orthogonal.

The eigenvectors are

$$\vec{v}_1=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad\vec{v}_2=\begin{bmatrix}-1\\0\\2\end{bmatrix},\quad\vec{v}_3=\begin{bmatrix}2\\0\\1\end{bmatrix}$$

Dividing by the norm gives us an orthonormal set and thus the required matrices are

$$D=\begin{bmatrix}-2&0&0\\0&4&0\\0&0&-1\end{bmatrix},\quad P=\begin{bmatrix}0&-1/\sqrt{5}&2/\sqrt{5}\\1&0&0\\0&2/\sqrt{5}&1/\sqrt{5}\end{bmatrix}$$

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