

Problem Set 2

Due September 20th at 4 pm in room 2-285.

Hand in parts 1 and 2 separately. Put your name on each part.

Part 1

1. Problem 15 from page 23.
2. (a) Give an alternate proof that $|x \cdot y| \leq |x||y|$ for any $x, y \in \mathbb{R}^n$ and for any $n \in \mathbb{N}$ by taking the inner product of the vector $|x|y - |y|x$ with itself and using the fact that this is nonnegative.
(b) Let a_1, \dots, a_n be positive real numbers. Prove that if

$$(a_1 + \dots + a_n) \left(\frac{1}{a_1} + \dots + \frac{1}{a_n} \right) \leq M$$

for some $M > 0$, then $n \leq \sqrt{M}$. When does equality hold?

3. Prove that a set is infinite in the sense of §2.4 if and only if it is in bijection with a proper subset of itself¹ by proving the following statements:
 - (a) If X is a proper subset of J_n for some $n \in \mathbb{N}$, then either X is empty or X is in bijection with J_k for some $k \in \mathbb{N}$, $k < n$.
 - (b) If $f: J_k \rightarrow J_\ell$ is an injection, then $k \leq \ell$. Moreover, $k = \ell$ if and only if f is also a surjection.
 - (c) No finite set can be in bijection with a proper subset of itself.
 - (d) \mathbb{N} is infinite.
 - (e) If X is infinite, there is an injection $f: \mathbb{N} \rightarrow X$.

Finally, use (c), (d), and (e) to conclude.

¹“The solution of the difficulties which formerly surrounded the mathematical infinite is probably the greatest achievement of which our own age has to boast. Since the beginnings of Greek thought these difficulties have been known; in every age the finest intellects have vainly endeavoured to answer the apparently unanswerable questions that had been asked by Zeno the Eleatic. At last Georg Cantor has found the answer, and has conquered for the intellect a new and vast province which had been given over to Chaos and old Night. It was assumed as self-evident, until Cantor and Dedekind established the opposite, that if, from any collection of things, some were taken away, the number of things left must always be less than the original number of things. This assumption, as a matter of fact, holds only of finite collections; and the rejection of it, where the infinite is concerned, has been shown to remove all the difficulties that had hitherto baffled human reason in this matter, and to render possible the creation of an exact science of the infinite.” From *The Study of Mathematics* by Bertrand Russell. <http://www.gutenberg.org/files/25447/25447-h/25447-h.htm>.

Part 2

4. Let S be a nonempty ordered set with the property that every nonempty subset of S has both a least upper bound in S and a greatest lower bound in S . Prove that if $f: S \rightarrow S$ satisfies $f(x) \leq f(y)$ for any $x, y \in S$ with $x \leq y$, then there exists an $x_0 \in S$ for which $f(x_0) = x_0$.
5. Problem 2 from page 43. You may use without proof the fact that if a_0, \dots, a_n are integers which are not all zero, then there are at most n complex solutions to $a_0z^n + a_1z^{n-1} + \dots + a_n = 0$.
6. Problem 9 from page 43.