Continuity

1. **Continuous functions.** We say that a function f taking real numbers to real numbers is *continuous* at s when either of the following equivalent conditions holds:

(1) For any sequence $s_n \to s$, we have $f(s_n) \to f(s)$.
(2) For any $\varepsilon > 0$, there is $\delta > 0$ such that $|x - s| < \delta$ implies $|f(x) - f(s)| < \varepsilon$.

The first condition says that f behaves nicely with respect to limits, so we can write

$$\lim_{n\to\infty} f(s_n) = f\Big(\lim_{n\to\infty} s_n\Big).$$

The second condition states that a small variation in the input leads to a small variation in the output. For instance suppose a toy train is on a track like the one in the picture:

The blue dot represents the starting position of the train, and the track goes in a straight line over a small hill. Suppose there is friction on the flat part but not on the hill. We launch the train to the right with some starting initial velocity, let it roll freely, and measure where it comes to a rest. Let f be the function taking the initial velocity to the final position. For small values of initial velocity the train comes to a rest on the left side of the hill, and for large values on the right side of the hill. For most values the function is continuous, because if we want to achieve a final position $f(x)$ to within a given tolerance $\varepsilon > 0$ of a given final position $f(s)$, then we need only get the initial velocity x right to within a suitable corresponding tolerance δ. But, at the critical velocity v_c at which the train has just enough energy to make it up the hill, there is a discontinuity: if $v < v_c$, then $f(v)$ is to the left of the hill, and if $v > v_c$, then $f(v)$ is to the right of the hill.

See Figure 1 for an illustration of ε and δ.

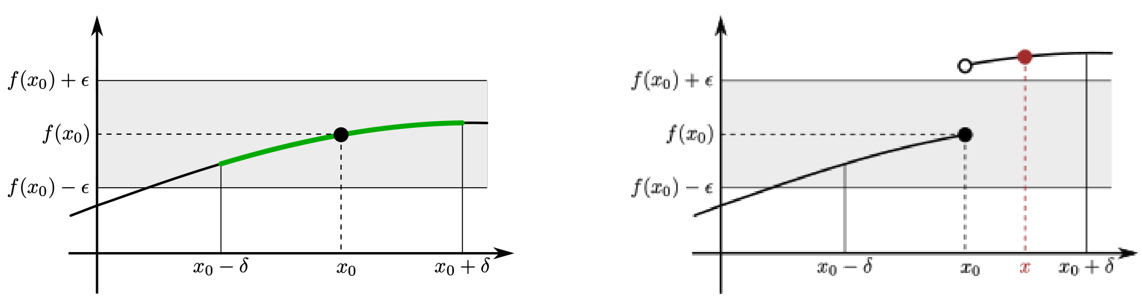


Figure 1. An illustration of definition (2) for a continuous function, and of its failure for a jump discontinuity.

Let us check that these two definitions are equivalent.

Kiril Datchev, February 5, 2026.

Proof. Suppose first that (2) holds. Let $s_n \to s$ be given and $\varepsilon > 0$ be given. By condition (2), there is $\delta > 0$ such that $|x - s| < \delta$ implies

$$|f(x) - f(s)| < \varepsilon. \tag{1.1}$$

Since $s_n \to s$, there is N such that $n > N$ implies $|s_n - s| < \delta$. Hence $n > N$ implies $|f(s_n) - f(s)| < \varepsilon$ by substituting $s_n = x$ into (1.1).

Now let us show that if (2) does not hold, then (1) does not hold either. If (2) does not hold, then there is $\varepsilon > 0$ such that, for any δ, we can find x such that $|x - s| < \delta$ and $|f(x) - f(s)| \geq \varepsilon$. This means that, for any n, we can find s_n such that $|s_n - s| < 1/n$ and $|f(s_n) - f(s)| \geq \varepsilon$. The inequality $|s_n - s| < 1/n$ shows that s_n converges to s, while the inequality $|f(s_n) - f(s)| \geq \varepsilon$ shows that $f(s_n)$ does not converge to $f(s)$. □

Example 1.2.

(1) The simplest examples of continuous functions are linear functions $f(x) = mx + b$. We check definition (1) using the limit laws of sequences: $s_n \to s$ implies that $ms_n + b \to ms + b$.
(2) Similarly, all rational functions are continuous wherever they are defined: if P and Q are polynomials, then the limit laws for sequences show that if $s_n \to s$ then $P(s_n)/Q(s_n) \to P(s)/Q(s)$, as long as the denominators are nonzero.
(3) Let us use definition (2) to check that the square root function is continuous at all nonnegative numbers. Let $\varepsilon > 0$ and $s > 0$ be given. If $|x - s| < \delta$, then, as long as $\delta < s$, we have

$$|\sqrt{x} - \sqrt{s}| = \frac{|x - s|}{\sqrt{x} + \sqrt{s}} < \frac{\delta}{\sqrt{s - \delta} + \sqrt{s}} < \frac{\delta}{\sqrt{s}}.$$

Thus we may take δ to be any number less than both s and $\varepsilon\sqrt{s}$. If $s = 0$, then the proof is easier because $0 \leq \sqrt{x} < \varepsilon$ as long as $0 \leq x < \varepsilon^2$.

Exercise 1.3.

(1) Find $\delta > 0$ such that $|x - 4| < \delta$ implies $|\sqrt{x} - 2| < 0.01$.
(2) Find $\delta > 0$ such that $|x - \frac{1}{2}| < \delta$ implies $|\frac{1}{x} - 2| < 0.01$.
(3) Show that the cube root function is continuous at all real numbers.[1]

We analogously have two equivalent definitions for the limit of a function of a real variable. We say that $f(x) \to L$ as $x \to a$, or that $\lim_{x \to a} f(x) = L$, when either

(1) For any sequence $s_n \to s$ such that $s_n \neq s$, we have $f(s_n) \to L$.
(2) For any $\varepsilon > 0$, there is $\delta > 0$ such that $0 < |x - s| < \delta$ implies $|f(x) - f(s)| < \varepsilon$.

Note the stricter requirement that $s_n \neq s$, or that $0 < |x - s|$; this corresponds to the fact that the limit is not affected by the value of the function at s (and indeed, the function need not be defined at s.) One can similarly formulate versions where one or both of a and L is infinite. We will do just one of these. We say that $f(x) \to \infty$ as $x \to -\infty$, when either

(1) For any sequence $s_n \to -\infty$, we have $f(s_n) \to \infty$.
(2) For any real number K, there is a real number M such that $x < M$ implies $f(x) > K$.

Exercise 1.4.

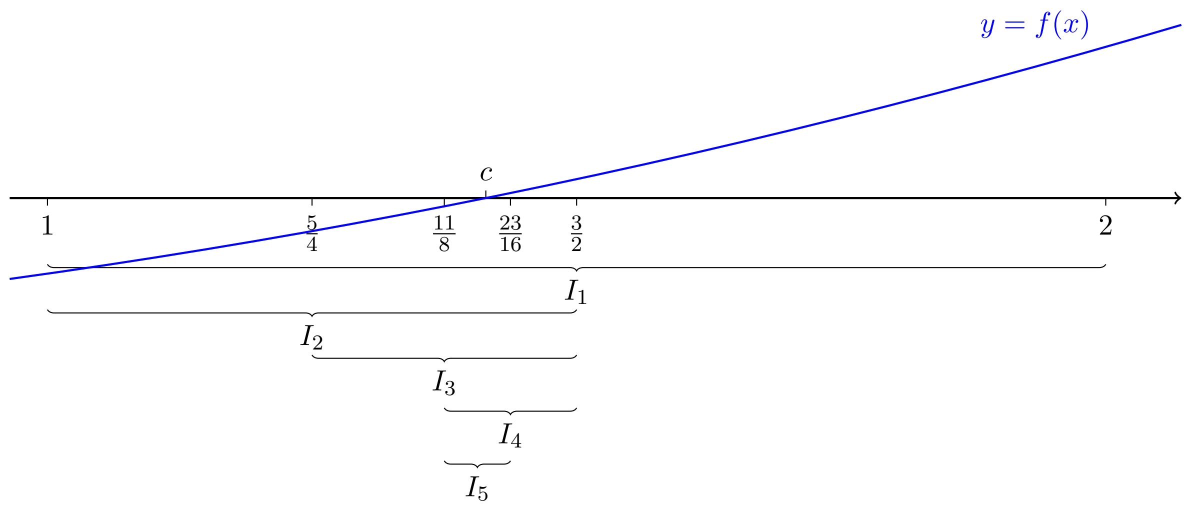
[1] *Hint:* Mimic the proof for the square root function, replacing the step $\sqrt{x} - \sqrt{s} = \frac{x-s}{\sqrt{x}+\sqrt{s}}$ by a similar step based on $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

(1) Formulate the corresponding equivalent definitions for $f(x) \to -\infty$ as $x \to s$, and prove that they are equivalent.
(2) Use one of the definitions to prove that $x/(x^2-1)^2 \to -\infty$ as $x \to 1$.

2. **Intermediate value theorem.** Continuity makes it possible to solve equations by bisection, generalizing the approach we already used for $\sqrt{2}$.

THEOREM 2.1 (Intermediate value theorem). *Let I be a closed interval and let $f\colon I \to \mathbb{R}$ be continuous.*[2] *If f has opposite signs at the endpoints of I, then there is c in I such that $f(c) = 0$.*

Proof. Let $I_1 = I$, and m_1 be the midpoint of I. If $f(m_1) = 0$ then we are done. If not, either the left half or the right half of I is an interval such that f has opposite signs at the enpoints. Let that half be I_2, and let m_2 be the midpoint of I_2. Continue this process to obtain a sequence of nested intervals $I_1 \supset I_2 \supset \cdots$, each half as long as the previous. Here is a picture for $f(x) = x^2 - 2$, building on the corresponding picture from the Introduction.



Then

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots \le \cdots b_n \le \cdots \le b_2 \le b_1.$$

We now put $c = \lim a_n = \lim b_n$, where a_n and b_n are the left and right endpoints of I_n. The limits exist because both sequences are monotone and bounded, and we prove that the limits agree by taking the limit of both sides of the equation $b_n - a_n = (2^{1-n})(b_1 - a_1)$. Moreover, by continuity, $\lim f(a_n) = \lim f(b_n) = f(c)$.

It remains to check that $f(c) = 0$. This follows from the fact that either $f(a_n) < 0 < f(b_n)$ for all n, or else $f(a_n) > 0 > f(b_n)$ for all n. Combining the inequalities with $\lim f(a_n) = \lim f(b_n) = f(c)$ yields $0 \le f(c) \le 0$ and hence $f(c) = 0$. $\square$

It follows that nth roots of positive numbers exist: apply Theorem 2.1 to $f(x) = x^n - q$.

[2] The notation $f\colon I \to \mathbb{R}$ means that f is a function which takes I to real numbers.

The intermediate value theorem has many fun applications to puzzles.[3] We start with a road trip. Let's say we drive from our home to our vacation on day 1, and then at the end of the vacation drive back home on day N. Suppose we take the same route there and back, but do not necessarily leave at the same time, make the same stops, or go at the same speeds. Is there a time at which we are at the same place on both days?

By the intermediate value theorem, the answer is yes. To see this, measure time in units of days, and let $d_1(t)$ be our distance from home at time t on day 1, and $d_N(t)$ our distance from home at time t on day N. Then $d_1(0) = d_N(1) = 0$, while $d_1(1) = d_N(0) = D$, where D is the distance between home and the vacation. Let $f(t) = d_1(t) - d_N(t)$. Then $f(0) = -D$, while $f(1) = D$, so there is c in $(0, 1)$ such that $f(c) = 0$, i.e. $d_1(c) = d_N(c)$, and at this time c we are in the same place on both days.

Another application is to wobbly tables. Suppose we have a symmetric square table, resting on the ground in such a way that two legs are stable but two are wobbling due to unevenness in the floor. We will show that the table can be stabilized, provided we can slide it around the floor in such a way that the wobble varies continuously.

To see this, observe that the table can only wobble with respect to opposite corners, and only at most one pair of opposite corners can wobble for any positioning of the table. In other words, either the pair AC wobbles and BD is stable, or vice versa, or both are stable. Suppose to begin with AC is wobbling and BD is stable. If we rotate the table by a right angle, then BD will be wobbling and AC will be stable.

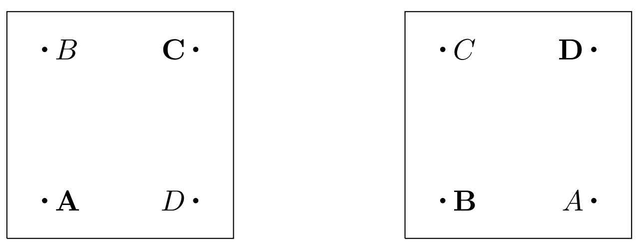


FIGURE 2. The table in its original and rotated positions. The bolded points are the wobbly ones.

Since wobbling varies continuously, and both pairs cannot wobble simultaneously, as we slide the table gradually from the original position to the rotated position, we must pass through a position where neither pair is wobbling.

More precisely, suppose the sliding takes place over a time interval $[0, 1]$, so that at $t = 0$ the table is in its original position and at $t = 1$ it is in its rotated position. Let $h_{AC}(t)$ and $h_{BD}(t)$ be the amplitudes of wobbling at each t, so that $h_{AC}(0) > 0$, $h_{BD}(0) = 0$, $h_{AC}(1) = 0$, $h_{BD}(1) = h_{AC}(0)$. Let $f(t) = h_{AC}(t) - h_{BD}(t)$. Then $f(0) = -f(1)$. Hence, by Theorem 2.1, there is c in the interval $(0, 1)$ such that $f(c) = 0$. That makes $h_{AC}(c) = h_{BD}(c) = 0$.

EXERCISE 2.2. Prove that $\cos(x) = x$ has a unique solution, using the fact that cosine is even, takes values in $[-1, 1]$, and is strictly decreasing in $[0, \pi]$.[4]

[3]See Section 4.5 of Loya's *Amazing and Aesthetic Aspects of Analysis* for all these and more.

[4]Incidentally, the iteration $s_{n+1} = \cos(s_n)$ converges to this solution for any initial value, but this will be easier to prove later with more technique.

3. **Extreme value theorem.** Physical processes settle to equilibrium states of minimal energy. Light rays take the path of least time. All motion follows principles of least action and their generalizations. Euler explains: the structure of the universe being most perfect, nothing happens anywhere without some rule of maximum or minimum shining forth.[5]

The extreme value theorem guarantees the *existence* of a maximizer and a minimizer of a continous function f over a compact interval $[a, b]$.

EXAMPLE 3.1. Given $\ell > 0$, what are the maximum and minimum area of a rectangle with perimeter 4ℓ, and what rectangles achieve these values? Let $f(x) = x(2\ell - x)$, the area of an x by $2\ell - x$ rectangle. Then $f(x)$ is maximized when $x = \ell$ and the rectangle is a square. But if x ranges over the open interval $(0, 2\ell)$, covering all possible rectangles of perimeter 4ℓ, then f can be an arbitrarily small positive number, but never zero. To get a minimum we extend f to the compact interval $[0, 2\ell]$, including the endpoints, and thus allowing generalized rectangles whose breadth may vanish. Then a minimal area $f(x) = 0$ occurs at $x = 0$ and $x = 2\ell$.

We now state and prove the general result.

THEOREM 3.2. *Let $f\colon [a, b] \to \mathbb{R}$ be continuous. Then there exists x^* in $[a, b]$ such that $f(x^*) \geq f(x)$ for all $x \in [a, b]$.*

This statement establishes the existence of a maximum. The existence of a minimum follows directly by applying the statement with $-f$ in place of f.

Proof. As in the proof of the intermediate value theorem, we use bisection: the interval $[a, b]$ is split in half, an appropriate one of the two halves is chosen, the chosen half is split in half, one of the two new halves is chosen, and so on, until in the limit only one point is left and this is the one we

More precisely, we say that the left half $[a, \frac{a+b}{2}]$ is *good* if, for any y in $[\frac{a+b}{2}, b]$, there is x in $[a, \frac{a+b}{2}]$ such that $f(x) \geq f(y)$. (Note that x is allowed to depend on y here.) If the left half is good, then let $I_1 = [a, \frac{a+b}{2}]$ and $J_1 = [\frac{a+b}{2}, b]$. If the left half is not good, then the right half is good, and we let $I_1 = [\frac{a+b}{2}, b]$ and $J_1 = [a, \frac{a+b}{2}]$. We call I_1 the first good interval, and J_1 the first rejected interval.

Now repeat the process with I_1 in place of $[a, b]$, to obtain a second good interval I_2 and a second rejected interval J_2, and iterate. Denoting the endpoints of I_k by $[a_k, b_k]$, we obtain the sequences

$$a_1 \leq a_2 \leq \cdots \leq a_k \leq \cdots \leq b_k \leq \cdots \leq b_2 \leq b_1, \quad b_k - a_k = (b - a)/2^k.$$

Hence there is a unique point x^* in each I_k, and $x^* = \lim a_k = \lim b_k$.

To be continued....

□

[5] *A method for discovering curves with the properties of maxima and minima, or solution of isoperimetric problems in the broadest accepted sense*, 1744.