Differentiation and Integration

1. **Differentiation.** The best way to locate a maximum or minimum of a function is often to check where the tangent to its graph is horizontal. The slope of the tangent of f at p is called the *derivative* of f at p and is defined by

$$f'(p) = \lim_{x\to p} \frac{f(x) - f(p)}{x - p}, \tag{1.1}$$

provided this limit exists.

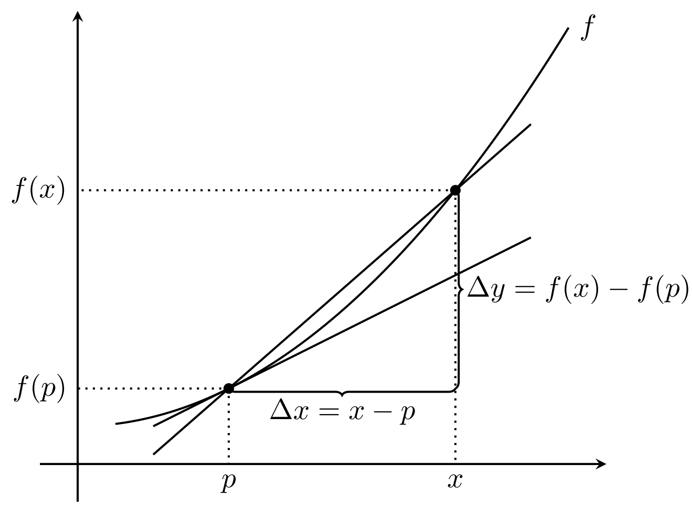

FIGURE 1. The derivative is the slope of the tangent line, obtained by taking the limit as $x \to p$ of the slopes of the secant lines $\Delta y/\Delta x$.

A function whose derivative exists is called *differentiable*. The following is the *derivative test* for finding a maximum:

THEOREM 1.2. *If $f\colon (a,b) \to \mathbb{R}$, x^* in (a,b), and $f(x^*) \ge f(x)$ for all x in (a,b), then either f is not differentiable at x^* or $f'(x^*) = 0$.*

Proof. Since $f(x^*) \ge f(x)$ for all x in (a,b), if $x > x^*$, then

$$\frac{f(x) - f(x^*)}{x - x^*} \le 0. \tag{1.3}$$

Similarly, if $x < x^*$, then

$$\frac{f(x) - f(x^*)}{x - x^*} \ge 0. \tag{1.4}$$

The two inequalities (1.3) and (1.4) show respectively that

$$\lim_{\substack{x\to x^* \\ x>x^*}} \frac{f(x) - f(x^*)}{x - x^*} \le 0, \qquad \lim_{\substack{x\to x^* \\ x<x^*}} \frac{f(x) - f(x^*)}{x - x^*} \ge 0,$$

Kiril Datchev, February 24, 2026.

i.e. the limit from the right is ≤ 0 and the limit from the left is ≥ 0. But if f is differentiable at x^*, then the limit from the left and the limit from the right are both equal to $f'(x^*)$, so the value of this limit is 0, i.e. $f'(x^*) = 0$. □

Exercise 1.5. State and prove the first derivative test for finding a minimum.

In addition to the *slope of the tangent line* definition above, derivatives also have an equivalent definition in terms of *linear approximation*. This is the next step up from constant approximation, which corresponds to continuity. Later we will use repeated differentiation to define quadratic, cubic, and higher approximations.

A function f is continuous at p if and only if

$$f(x) = f(p) + r_0(x), \tag{1.6}$$

where r_0 is a remainder that goes to zero at p, i.e. $r_0(x) \to 0$ as $x \to p$. The constant function $x \mapsto f(p)$ is the *constant approximation* to f at p. It is also known as the zeroth order Taylor polynomial of f at p.

A function f is differentiable at p with derivative $f'(p)$ if and only if

$$f(x) = f(p) + f'(p)(x-p) + r_1(x), \tag{1.7}$$

where r_1 is a remainder that goes to zero at p faster than linearly, i.e. $\dfrac{r_1(x)}{x-p} \to 0$ as $x \to p$. The linear function $x \mapsto f(p) + f'(p)(x-p)$ is the *linear approximation* to f at p. It is also known as the first order Taylor polynomial of f at p.

Proof of equivalence of (1.1) *and* (1.7). Suppose first that f is differentiable at p with derivative $f'(p)$. Let $r_1(x) = f(x) - f(p) - f'(p)(x-p)$. Then

$$\frac{r_1(x)}{x-p} = \frac{f(x)-f(p)}{x-p} - f'(p) \to 0, \qquad \text{as } x \to p.$$

Hence (1.7) holds with $r_1(x)/(x-p) \to 0$ as $x \to p$.

Suppose conversely that (1.7) holds with $r_1(x)/(x-p) \to 0$ as $x \to p$. Then

$$\frac{f(x)-f(p)}{x-p} = f'(p) + \frac{r_1(x)}{x-p} \to f'(p), \qquad \text{as } x \to p.$$

Hence f is differentiable at p with derivative $f'(p)$. □

Exercise 1.8.

(1) Mimic the proof above to show that f is continuous at p if and only if the remainder r_0 in (1.6) obeys $r_0(x) \to 0$ as $x \to p$.
(2) Use (1.6) and (1.7) to show that if a function is differentiable at p, then it is continuous at p.
(3) Factor the polynomials $r_0(x)$ and $r_1(x)$ in the case $f(x) = x^2$, $p = 1$, $f'(p) = 2$. Use this factorization to evaluate the limits of $r_0(x)$ and $r_1(x)/(x-p)$ as $x \to p$.

EXERCISE 1.9. (Challenging!) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that f' has no jump discontinuities[1] by following these steps:

(1) Suppose for contradiction that f' has a jump discontinuity at some point p. Show that there is then an interval $[a, b]$ such that p is in (a, b) and such that f' omits a value c between the values of the limit from left and right of f' at p.
(2) Prove that $f - c$ attains a maximum or minimum at some point s in (a, b), and then use the derivative test for a maximum or minimum to conclude that $f'(s) = c$, achieving the desired contradiction.

2. **Rules of differentiation.** It is easy to compute the derivative of a linear function: if $f(x) = mx+b$, then $f'(x) = m$.

EXERCISE 2.1. Show that if $f(x) = mx + b$, then $f'(x) = m$, in two ways: once using (1.1) and once using (1.7).

Powerful rules of differentiation help us deal with derivatives of more complicated functions. To prove them, it is helpful to write (1.7) as

$$f(x) = f(p) + f'(p)(x - p) + r(x; f),$$

making explicit the dependence of the remainder on the function f.

THEOREM 2.2. *If f and g are differentiable at p, then so are the sum $f+g$ and product fg. If $g(p) \neq 0$ then so is the quotient f/g. Moreover, we have the sum rule, product rule, and quotient rule:*

$$(f + g)' = f' + g', \qquad (fg)' = f'g + fg', \qquad \frac{f}{g} = \frac{f'g - fg'}{g^2},$$

where all functions are evaluated at p.

Proof. For the product rule, write

$$\begin{aligned} f(x)g(x) &= \Big(f(p) + f'(p)(x - p) + r(x; f)\Big)\Big(g(p) + g'(p)(x - p) + r(x; g)\Big) \\ &= f(p)g(p) + \Big(f'(p)g(p) + f(p)g'(p)\Big)(x - p) + r(x, fg), \end{aligned}$$

where $r(x, fg) = f(p)r(x; g)+r(x; f)g(p)+r(x; f)r(x; g)$. We only need to check that $r(x, fg)/(x-p) \to 0$ as $x \to p$. For this, write

$$\frac{r(x, fg)}{x - p} = f(p)\frac{r(x, g)}{x - p} + \frac{r(x, f)}{x - p}g(p) + \frac{r(x, f)}{x - p}r(x, g),$$

and observe that each term on the right goes to zero as $x \to p$ because it is the product of a quotient which goes to zero and a function which is either constant or goes to zero as $x \to p$.

The reciprocal rule is the case $f = 1$ of the quotient rule. we prove it using (1.1):

$$\frac{\frac{1}{g(x)} - \frac{1}{g(p)}}{x - p} = \frac{1}{x - p}\,\frac{g(p) - g(x)}{g(x)g(p)} \to -\frac{g'(p)}{g(p)^2}.$$

□

[1]A *jump discontinuity* of a function g is a point p such that the limit from the left $\lim_{\substack{x \to p \\ x < p}} g(x)$ and the limit from the right $\lim_{\substack{x \to p \\ x > p}} g(x)$ both exist but have different values.

Exercise 2.3.

(1) Prove the sum rule and quotient rule; the former is an easier version of the proof of the product rule, and the latter can be deduced from the product rule and reciprocal rule, or can be proven from scratch following similar steps to the proof of the reciprocal rule above.
(2) Use the result of Exercise 2.1, the product rule, induction, and the reciprocal rule, to show that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$, for any integer n.

The chain rule deals with composite functions, which we write as $(g \circ f)(x) = g(f(x))$.

Theorem 2.4. *If f is differentiable at p and g is differentiable at $q = f(p)$, then the composition $h = g \circ f$ is differentiable at p, and*

$$h'(p) = g'(q)f'(p).$$

Proof. The cleanest calculation is to write

$$\begin{aligned}\lim_{x\to p} \frac{g(f(x)) - g(f(p))}{x-p} &= \lim_{x\to p} \frac{g(f(x)) - g(f(p))}{f(x)-f(p)} \frac{f(x)-f(p)}{x-p} \\ &= \lim_{y\to q} \frac{g(y)-g(q)}{y-q} \lim_{x\to p} \frac{f(x)-f(p)}{x-p} = g'(q)f'(p),\end{aligned}$$

where in the first step we use $f(x) \neq f(p)$, and in the second step use the fact that $y = f(x) \to q = f(p)$ when $x \to p$ because f is continuous. The problem with this is the requirement that $f(x) \neq f(p)$, which does not always hold and may be difficult to check even if it does hold.

To get around this, we use the same strategy as in the proof of the product rule, but using a different notation which is more convenient here. Write

$$f(x) = f(p) + (x-p)\Big(f'(p) + u(x)\Big), \qquad g(y) = g(q) + (y-q)\Big(g'(q) + v(y)\Big),$$

where $u(x) \to 0$ as $x \to p$, $v(y) \to 0$ as $y \to q$. Let $y = f(x)$. Then

$$\begin{aligned}g(y) &= g\Big(q + (x-p)\Big(f'(p) + u(x)\Big)\Big) = g(q) + g'(q)(x-p)\Big(f'(p) + u(x)\Big) + (y-q)v(y) \\ &= g(q) + (x-p)\Big(g'(q)f'(p) + w(x)\Big),\end{aligned}$$

where $w(x) = g'(q)u(x) + v(y)(y-q)/(x-p)$. Next,

$$w(x) = g'(q)u(x) + v(y)\frac{y-q}{x-p} \to 0, \qquad \text{as } x \to p,$$

because $y \to q$ as $x \to p$, and hence $u(x) \to 0$ and $v(y) \to 0$, and finally

$$\frac{y-q}{x-p} = \frac{f(x)-f(p)}{x-p} \to f'(p).$$

□

3. **Mean value theorem.** This is the big theorem for differentiable functions, analogous to the intermediate and extreme value theorems for continuous functions.

THEOREM 3.1. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous, and suppose f is differentiable on (a,b). Then there is c in (a,b) such that the slope of the tangent line through c equals the slope of the secant line through the endpoints:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

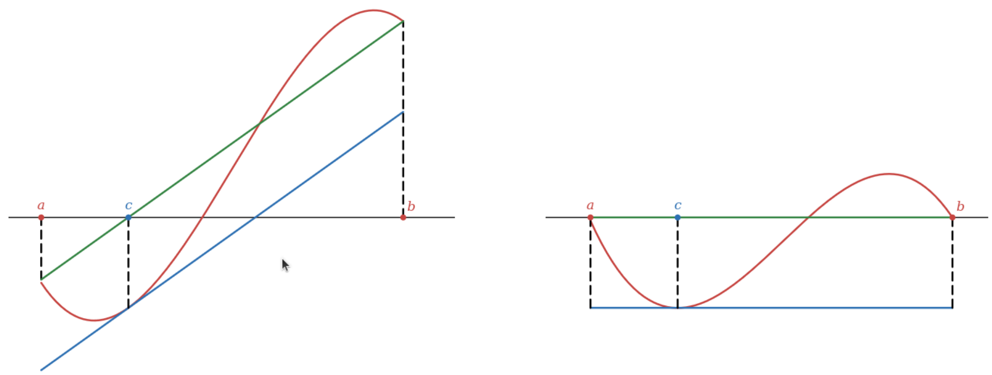

FIGURE 2. On the left, the function f from the mean value theorem, with the secant line in green and the tangent line in blue. On the right, the same thing for the function g from Rolle's theorem.

We begin with an easier special case, known as Rolle's theorem:

THEOREM 3.2. *Let $g\colon [a,b] \to \mathbb{R}$ be continuous, and suppose g is differentiable on (a,b) and $g(a) = g(b)$. Then there is c in (a,b) such that $g'(c) = 0$*

Proof of Theorem 3.2. By the extreme value theorem, g attains a maximum and a minimum on $[a,b]$. At least one of these must be attained on the interior (a,b), because if both are attained at the endpoints then both are equal (since $g(a) = g(b)$) and hence the function is constant. Let c be an interior max or min of g. Then $g'(c) = 0$ by the derivative test, Theorem 1.2. □

Proof of Theorem 3.1. Apply Theorem 3.2 to $g(x) = f(x) - mx$, where $m = (f(b) - f(a))/(b - a)$. This is justified because

$$g(b) = \frac{f(a)b - f(b)a}{b - a} = g(a),$$

and results in a point c in (a,b) such that $g'(c) = 0$, i.e $f'(c) = m$. □

The mean value theorem leads to simple characterizations of convex functions. A function $\mathbb{I} \to \mathbb{R}$, where I is an open interval, is called *convex* when the slopes of its secant lines are increasing:

$$a < x < b \qquad \Longrightarrow \qquad \frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(x)}{b - x}. \tag{3.3}$$

An equivalent condition is that its graph lies below its secant lines in the sense that:

$$a < x < b \qquad \Longrightarrow \qquad f(x) \leq \frac{f(b)(x - a) + f(a)(b - x)}{b - a}. \tag{3.4}$$

EXERCISE 3.5. Prove the equivalence of (3.3) and (3.4) algebraically.

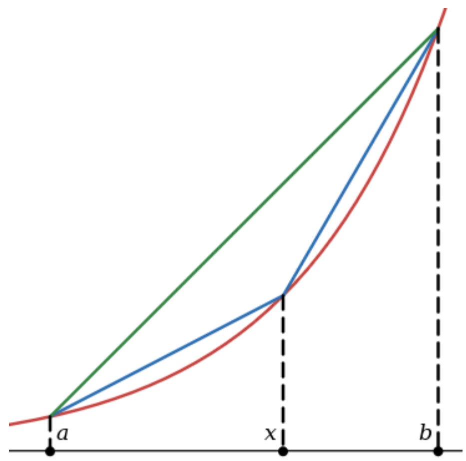

FIGURE 3. A convex function, with the secant lines from (3.3) in blue, and with the secant line from (3.4) in green.

THEOREM 3.6. *Let $f\colon I \to \mathbb{R}$ be differentiable. Then* (3.3) *is equivalent to*

$$a < b \qquad \Longrightarrow \qquad f'(a) \leq f'(b). \tag{3.7}$$

Proof. Starting with (3.7), we derive (3.3) by using the mean value theorem to conclude that there are c in (a, x) and d in (x, b) such that

$$\frac{f(x) - f(a)}{x - a} = f'(c), \qquad \frac{f(b) - f(x)}{b - x} = f'(d).$$

Since $c < d$, (3.7) implies that $f(c) \leq f(d)$, which yields (3.3).

Conversely, starting with (3.3), we derive (3.7) by taking the limit once as $x \to a$ and once as $x \to b$. These yield respectively

$$f'(a) \leq \frac{f(b) - f(a)}{b - a}, \qquad \frac{f(b) - f(a)}{b - a} \leq f'(b).$$

Since this works for any $a < b$, combining the inequalities yields (3.7). □

One of the most important properties of a convex function is that any local minimum is a global minimum. In the differentiable case, this follows from the fact that the derivative vanishes at a local minimum, combined with the monotonicity of the derivative and the following theorem:

THEOREM 3.8. *Let $f\colon I \to \mathbb{R}$ be differentiable. The following are equivalent:*

(i) *If $a < b$, then $f(a) \leq f(b)$.*
(ii) *$f'(x) \geq 0$ for all x.*

EXERCISE 3.9. Prove Theorem 3.8 by mimicking the proof of Theorem 3.6.

EXERCISE 3.10.

(1) Prove that if $f\colon I \to \mathbb{R}$ is differentiable with $f'(x) > 0$ for all x, and if $a < b$, then $f(a) < f(b)$.

(2) Give an example of a polynomial f such that if $a < b$, then $f(a) < f(b)$, and yet it is not true that $f'(x) > 0$ for all x.

A crucial corollary of Theorem 3.8 for the theory of integration is the fact that if $f'(x) = 0$ for all x, then f is constant.

EXERCISE 3.11. Use Theorem 3.8 (or the mean value theorem directly) to show that if $f\colon I \to \mathbb{R}$ and $f'(x) = 0$ for all x in I, then f is constant.

It is what allows us to conclude that, if $F' = f$, then F is uniquely determined up to a constant. This may seem innocuous, but it uses completeness in a fundamental way: see the following exercises.

EXERCISE 3.12. Define $f\colon \mathbb{Q} \to \mathbb{Q}$ (recall that $\mathbb{Q}$ denotes the set of rational numbers) by

$$f(x) = \begin{cases} 1, & x^2 > 2 \\ 0, & \text{otherwise.} \end{cases}$$

Prove that $f'(x) = 0$ for all x (note that x is restricted to being rational!) even though f is not constant. Use in your proof the fact that every interval of real numbers with a rational endpoint contains rational numbers in its interior.

EXERCISE 3.13. The proof above, that f is a constant whenever $f'(x) = 0$ for all x, has the following structure: it relies on the mean value theorem, which relies on the extreme value theorem, which has a complicated proof. Here is a more direct proof:

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(b) > f(a)$ for some $a < b$, and let $m = (f(b) - f(a))/(b - a)$ be the slope of the secant line over $[a, b]$. Consider the two halves $[a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b]$. Prove algebraically that the slope of the secant line over at least one of these halves is $\geq m$. Apply the method of bisection, at each step choosing a half with slope $\geq m$, to obtain a point in $[a, b]$ at which the derivative is $\geq m$.
(2) Use the result of part (1) to conclude that if $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function whose derivative is always zero, then f is a constant.

4. **Integration.**

The *integral* of a positive function f over an interval $[a, b]$, written

$$\int_a^b f = \int_a^b f(t)\, dt,$$

is the area underneath its graph. For general f this notion involves difficulties, but if f is constant we have simply

$$\int_a^b c = (b - a)c. \tag{4.1}$$

The formula (4.1) makes it natural to allow a, b, and c to be arbitrary real numbers, upon which the meaning of the integral becomes a signed area, with going below the real axis or going from right to left counting as negative.

The problem of rigorously defining an integral, or of defining the area under the graph of a function, is notoriously complicated. The simplest approach leads to the Riemann or Darboux integral (these are two names for the same object) while for more advanced work one eventually needs the more powerful Lebesgue integral.

We will ultimately use the Riemann integral below, but we begin by introducing the main general properties of all integrals, which are gathered in Theorem 4.5 below. The point of this approach is to emphasize these general properties, and deemphasize what is specific to Riemann integrals.

The *fundamental theorem of calculus* states that differentiation and integration are inverse operations. Specifically, if f is continuous at p, then

$$\begin{aligned}\frac{d}{dp}\int_a^p f &= \lim_{h\to 0}\frac{\int_a^{p+h} f - \int_a^p f}{h} = \lim_{h\to 0}\frac{1}{h}\int_p^{p+h} f(x)\,dx = \lim_{h\to 0}\frac{1}{h}\int_p^{p+h} f(p)\,dt = \lim_{h\to 0}\frac{1}{h}hf(p) \\ &= f(p).\end{aligned} \tag{4.2}$$

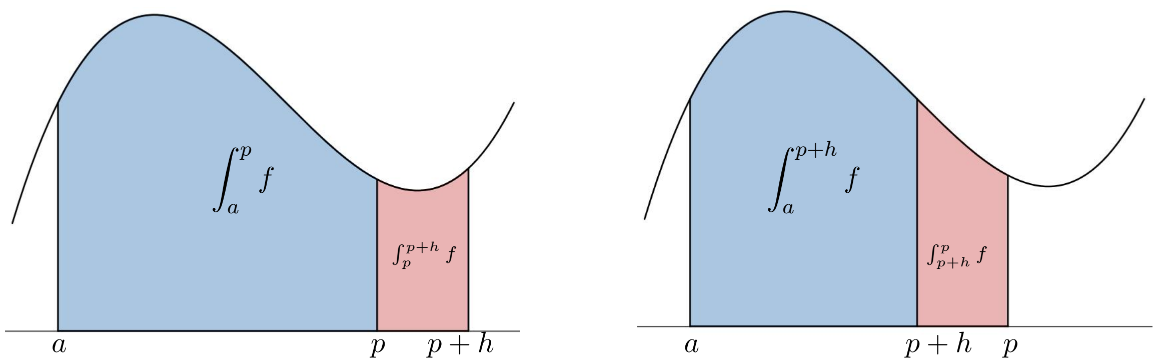

FIGURE 4. The areas used in the calculation of the fundamental theorem of calculus, when $a < p$, in the cases $h > 0$ and $h < 0$.

Let us examine the steps of the calculation (4.2), from simplest to most complicated.

The first equality is the definition of the derivative. The second uses

$$\int_a^b f + \int_b^c f = \int_a^c f,$$

and the fourth uses (4.1).

The third is the hardest. By definition of continuity, given any $\varepsilon > 0$, there is $\delta > 0$ such that $|p - x| \le \delta$ implies $|f(p) - f(x)| \le \varepsilon$. In other words,

$$f(p) - \varepsilon \le f(x) \le f(p) + \varepsilon.$$

Consequently, if $0 < h \le \delta$, then

$$f(p) - \varepsilon = \frac{1}{h}\int_p^{p+h}(f(p) - \varepsilon)\,dx \le \frac{1}{h}\int_p^{p+h} f(x)\,dx \le \frac{1}{h}\int_p^{p+h}(f(p) + \varepsilon)\,dx = f(p) + \varepsilon.$$

This implies $|\frac{1}{h}\int_p^{p+h} f(x)\,dx - f(p)| \le \varepsilon$ when $0 < h \le \delta$, and hence

$$\lim_{\substack{h\to 0\\ h>0}}\frac{1}{h}\int_p^{p+h}\Big(f(p) - f(x)\Big)\,dx = 0, \qquad \text{or} \qquad \lim_{\substack{h\to 0\\ h>0}}\frac{1}{h}\int_p^{p+h} f(x)\,dx = \lim_{\substack{h\to 0\\ h>0}}\frac{1}{h}\int_p^{p+h} f(p)\,dx.$$

The treatment of the case $-\delta \le h < 0$ is similar:

EXERCISE 4.3. Use

$$\int_a^b f = -\int_b^a f \tag{4.4}$$

to show that

$$\lim_{\substack{h\to 0\\ h<0}} \frac{1}{h}\int_p^{p+h} f(x)\,dx = \lim_{\substack{h\to 0\\ h<0}} \frac{1}{h}\int_p^{p+h} f(p)\,dx.$$

In summary, we have proven the following:

Theorem 4.5 (Fundamental theorem of calculus). *Let $I \subset \mathbb{R}$ be an open interval, and f a function from I to $\mathbb{R}$. Suppose the integral $\int_a^b f$ is defined for all a and b in I in such a way that the following two properties hold:*

- *Integration is additive over intervals, in the sense that for any a, b, c in I we have*

$$\int_a^b f + \int_b^c f = \int_a^c f. \tag{4.6}$$

- *Integration respects bounds: if $a < b$ and $m \leq f(x) \leq M$ whenever $a \leq x \leq b$, then*

$$(b-a)m \leq \int_a^b f \leq (b-a)M. \tag{4.7}$$

Then

$$\frac{d}{dp}\int_a^p f = f(p),$$

whenever f is continuous[2] *at p.*

The properties (4.6) and (4.7) correspond to additivity properties of areas. Additivity over intervals is the fact that the total area of the red and blue regions in Figure 4 equals the sum of the area of the blue region and the area of the red region. See Figure 5 for (4.7). This makes (4.6) and (4.7) natural properties that any definition of integral should have. Note that (4.7) implies $\int_a^a f = 0$, and combining this with (4.6) shows that $\int_a^b f = -\int_b^a f$.

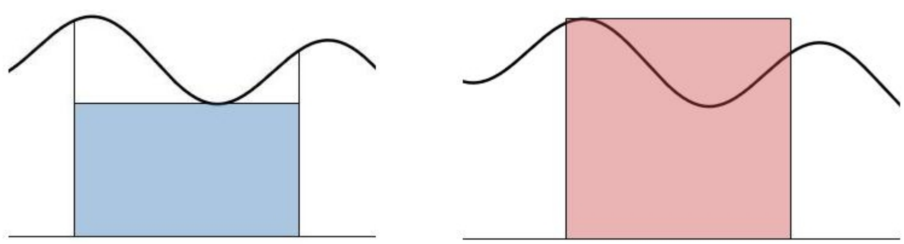

Figure 5. The area under the curve, between the vertical lines, is $\geq$ the blue area and $\leq$ the red area.

[2]When f is not continuous at p, we cannot expect $p \mapsto \int_a^p f$ to be differentiable at p, but we do have the following:

Exercise 4.8. Use (4.6) and (4.7) to prove that if $|f(x)| \leq K$ for all x in I, and a is in I, then the function $p \mapsto \int_a^p f$ is continuous.

Hint: Show that

$$\left|\int_a^{p+h} f - \int_a^p f\right| \leq K|h|.$$

The difficulty at this point is defining the integral appropriately. But for continuous functions with known antiderivatives we are in good shape.

THEOREM 4.9. *Let f, $F\colon I \to \mathbb{R}$ be such that $F' = f$ and f is continuous. Then the integral $\int_a^b f$ obeys* (4.6) *and* (4.7) *if and only if it is defined by*

$$\int_a^b f = F(b) - F(a). \tag{4.10}$$

Proof. Suppose first that $\int_a^b f$ is defined by (4.10). Then (4.6) follows by plugging in (4.10) and canceling. Next, for (4.7), observe that the mean value theorem implies there is c in (a, b) such that

$$\int_a^b f = F(b) - F(a) = F'(c)(b-a) = f(c)(b-a),$$

and use $m \le f(c) \le M$.

Now suppose that $\int_a^b f$ is defined in such a way that (4.6) and (4.7) both hold, and let $G(x) = F(x) - \int_a^x f$. By Theorem 4.5,

$$G'(x) = F'(x) - f(x) = 0,$$

and hence G is a constant by Exercise 3.13. Consequently,

$$\int_a^b f = F(b) - G(b) = F(b) - G(a) = F(b) - F(a),$$

where at the first step we used the definition of G, at the second the fact that G is constant, and at the third again the definition of G. □

If there is no known antiderivative (and many continuous functions have no simple antiderivative) then we must work harder, and directly define the integral so that the properties of additivity over integrals (4.6) and respect of bounds (4.7) hold. Our approach, which goes back to Riemann, will be to appoximate f using piecewise constant functions, whose integrals can be evaluated using (4.1). It is simplest to use fuctions which are either all above or all below our function: see Figure 6. For a continuous function it doesn't matter which are chosen, and we will choose the latter.

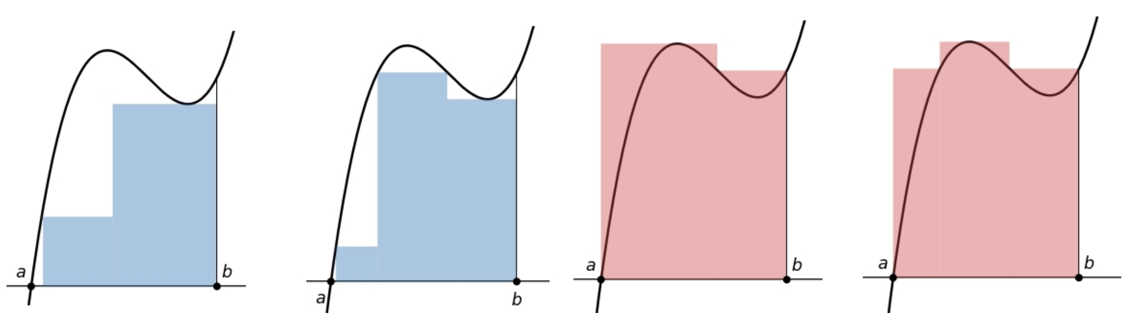

FIGURE 6. The blue piecewise constant functions approximate f from below; they correspond to φ from (4.11). The red ones do the same thing from above; they correspond to (4.12).

More preciesly, we put

$$\mathbf{L}\int_a^b f = \sup\left\{ \int_a^b \varphi \colon \varphi \text{ is a piecewise constant function such that } \varphi(x) \le f(x) \text{ for all } x \right\}. \tag{4.11}$$

Here $\sup A$ denotes the *supremum* or least upper bound of the set A; it is the smallest number K such that $a \le K$ for all a in A. We call this the *lower integral* because f is being approximated from below. We could just as well put

$$\mathbf{U}\int_a^b f = \inf\left\{\int_a^b \varphi\colon \varphi \text{ is a piecewise constant function such that } \varphi(x) \ge f(x) \text{ for all } x\right\}. \quad (4.12)$$

Here inf is the *infimum* or greatest lower bound, and this one is called the *upper integral.*

Example 4.13.

(1) Let $A = \{\frac{1}{2x^2+3} : x \in \mathbb{R}\}$. Then $A = (0, 1/3]$ and $\sup A = 1/3$, $\inf A = 0$. In this case the sup is an element of the set, and is also called a maximum. But the set has no minimum.
(2) Let $A = \{\frac{4}{5+6^n} : n \in \mathbb{Z}\}$, where $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. This time

$$\sup A = \lim_{n\to-\infty} \frac{4}{5+6^n} = \frac{4}{5}, \qquad \inf A = \lim_{n\to\infty} \frac{4}{5+6^n} = 0,$$

and neither one is in the set: the set has no maximum or minimum.

It is one of the crucial facts about real numbers that every nonempty set has a sup if it is bounded above and an inf if it is bounded below. It is enough to check this for sup, because $\inf A = \sup B$ when B is the set of x such that $-x$ is in A.

Exercise 4.14. Let A be a nonempty set of real numbers which is bounded above. Use the method of bisection to prove that there is a unique real number α, called the supremum of A, with the following two properties:

(i) If x is in A then $x \le \alpha$.
(ii) If $\beta < \alpha$ then there is x in A such that $x > \beta$.

The first property says that α is an upper bound, and the second says that no smaller number is an upper bound.[3]

Exercise 4.15. Let $A = \{\frac{2n}{3+4n} : n = 0, 1, 2, \dots\}$. Find $\sup A$, and prove in detail that the properties (i) and (ii) from Exercise 4.14 hold.

[3] *Hint:*Apply the method of bisection to an interval $[a, b]$, where b is any number that is an upper bound and a is any number that is not an upper bound.