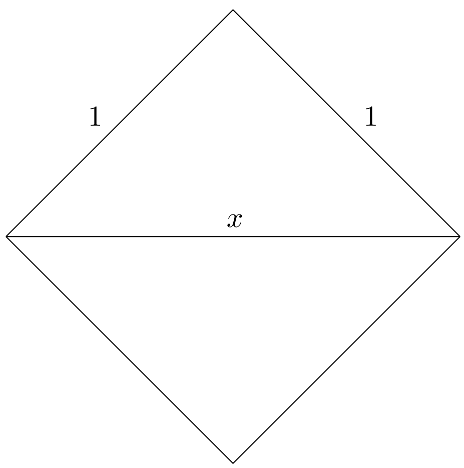
Introduction

1. **The square root of two.** The first analysis problem ever solved is finding the diagonal of a square in terms of the sides. By the Pythagorean theorem, this means solving $x^2 = 1^2 + 1^2 = 2$, or in other words computing $x = \sqrt{2}$. This is where we begin, in the footsteps of the four thousand year old clay tablet YBC 7289.



The writing across the horizontal diagonal is 1 24 51 10, which stands for $1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$. This sum computes $x = \sqrt{2}$ to within one part in two million. More precisely,

$$\frac{1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} - \sqrt{2}}{\sqrt{2}} = -0.000042\%. \tag{1.1}$$

For a square with side length one kilometer (greater than the height of any building), this computes the diagonal to accuracy better than half a millimeter (the diameter of a grain of sand). We will see that, with the right technique, it is not hard to recover and even improve this result.

Here *analysis* means *analysis of the infinite*, and the analysis on the tablet is only implicit. As Newton explains a few millenia later, after presenting his general method for solving equations with infinitely many terms [New, Section 52], the analysis of the infinite subsumes the analysis of the finite, and this is why it has become standard to just say *analysis*:

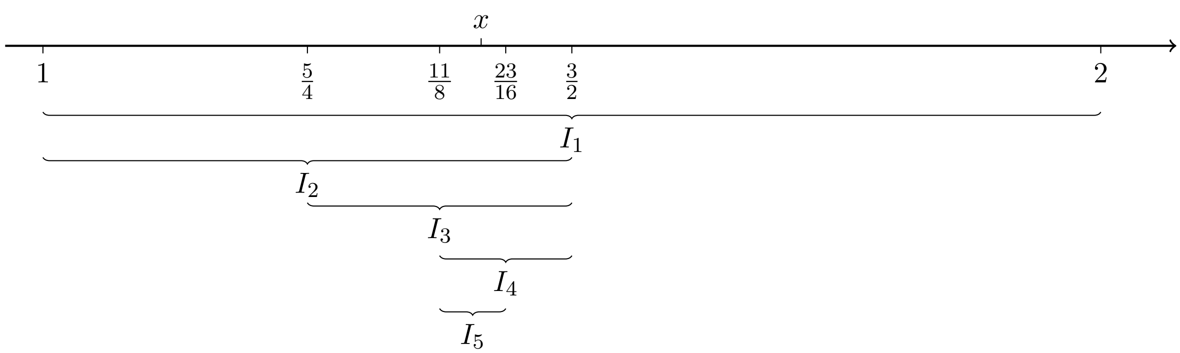
> Whatever common analysis performs by means of a finite number of terms (provided that can be done) this method can always perform the same by means of infinite equations, so that I have not made any question of giving it the name of *analysis* likewise.

For a more complete discussion of the development of analysis, see [Dat], especially cxviii and cccx for the tablet and clxviii for Newton's *Analysis by equations of an infinite number of terms.*

Kiril Datchev, January 13, 2026.

2. **Computing $\sqrt{2}$ by bisection.** The most straightforward approach to $\sqrt{2}$ is the crude but fundamental method of bisection, or binary search.

Let I_1 be the interval $(1, 2)$, and for each $n = 1, 2, 3, \ldots$, let I_{n+1} be the interval containing $\sqrt{2}$ which is half as long as I_n and shares an endpoint with I_n. Here is a number line with the sequence of intervals:



Explicitly, we obtain these intervals as follows. Since $1^2 < 2 < 2^2$, we know that $\sqrt{2}$ is in I_1. The midpoint of I_1 is $3/2$, which we denote m_1. Since $m_1^2 = 9/4 > 2$, we know that $\sqrt{2}$ is in $(1, 3/2)$, and so we take $I_2 = (1, 3/2)$. Similarly, $m_2 = 5/4$, $m_2^2 = 25/16$, and $I_3 = (5/4, 3/2)$. Continuing in this way, we obtain the following table.

n	1	2	3	4
I_n	$(1, 2)$	$\left(1, \frac{3}{2}\right)$	$\left(\frac{5}{4}, \frac{3}{2}\right)$	$\left(\frac{11}{8}, \frac{3}{2}\right)$
m_n	$\frac{3}{2} = 1.1$	$\frac{5}{4} = 1.01$	$\frac{11}{8} = 1.011$	$\frac{23}{16} = 1.0111$
m_n^2	$\frac{9}{4} = 2 + \frac{1}{4}$	$\frac{25}{16} = 2 - \frac{7}{16}$	$\frac{121}{64} = 2 - \frac{7}{64}$	$\frac{529}{256} = 2 + \frac{17}{256}$

In this table, m_n is expanded in binary, with 0s in the expansion corresponding to left subintervals, and 1s to right subintervals.

Since $\sqrt{2}$ is in I_n, the distance from $\sqrt{2}$ to the midpoint of I_n, is less than half the length of I_n, which equals $1/2^n$. This implies the following bound on the accuracy of the approximations m_n:

$$|\sqrt{2} - m_n| < 1/2^n. \tag{2.1}$$

Thus for example the distance from $\sqrt{2}$ to $5/4$ is less than $1/4$, and the distance from $\sqrt{2}$ to $23/16$ is less than $1/16$. Sometimes we do get lucky: as we see from the last row of the table, the accuracy of m_1 is actually better than that of m_2. Nevertheless, comparing (1.1) and (2.1) shows that, to obtain the result of YBC 7289, unless we are very lucky, we must either go up to a rather large value of n, or find a better method.

Exercise 2.2. Add a column to the table above for $n = 5$, and a row with $a_n = 1/2^n$ for the accuracy bound.[1] For which value of n does $1/2^n$ reproduce the accuracy of YBC 7289?

3. **What kind of number is $\sqrt{2}$?** Of course the method of bisection does sometimes get lucky, and m_n produces much better accuracy than the bound $a_n = 1/2^n$ from (2.1) suggests. For example, at the stage $n = 1$ it might seem that $m_1 = 3/2$ is only accurate to within $a_1 = 1/2$, but from the result of stage $n = 4$ we can see that $\sqrt{2}$ is in $I_4 = (11/8, 3/2)$, and hence $\sqrt{2} \approx 3/2$ is accurate to within $a_3 = 1/8$.

Might we get completely lucky by this method, and find n such that $m_n = \sqrt{2}$ exactly? The answer is no, because m_n is rational and $\sqrt{2}$ is not.

To explain this, we begin by recalling the fact that any rational number can be written in *lowest terms*: this means writing it so that the denominator is as small as possible. For example, 8/12 in lowest terms is 2/3, and the solution to $4t + 7 = 9$ in lowest terms is $t = 1/2$.

We now proceed with the classic Pythagoarean proof by contradiction that $x^2 = 2$ has no rational solutions. Suppose for contradiction that $(m/n)^2 = 2$, and that m/n is in lowest terms. Then

$$\frac{m^2}{n^2} = 2, \text{ which implies } m^2 = 2n^2.$$

Hence m is even[2] and we can write $m = 2m'$, where m' is an integer. Then

$$(2m')^2 = 2n^2, \text{ which implies } 2m'^2 = n^2.$$

Hence n is even and we can write $n = 2n'$, where n' is an integer. Thus

$$\frac{m}{n} = \frac{2m'}{2n'} = \frac{m'}{n'}.$$

Since $n' = n/2 < n$ this is in contradiction with the fact that m/n is in lowest terms. Consequently, there is no rational number x such that $x^2 = 2$.

This proof has far-reaching variants and generalizations: see [HaWr, Chapter IV]. An interesting one, which avoids factoring, is the following, from [Ded, Section IV].

Exercise 3.1. Suppose q is a positive integer which is not the square of an integer (in other words $q \neq 1$, $q \neq 4$, $q \neq 9$, etc.). Let us show that $x^2 = q$ has no rational solutions. Suppose for contradiction that $(m/n)^2 = q$, and that m/n is in lowest terms. Let k be the positive integer such that $k^2 < q < (k+1)^2$. Let $n' = m - nk$ and let $m' = nq - mk$. Prove that[3] $0 < n' < n$, and that $m'^2 - qn'^2 = 0$. Conclude that $(m'/n')^2 = q$, contradicting the minimality of n.

4. **Computing $\sqrt{2}$ by recursive averages.** We wish to define a sequence $x_1, x_2, \ldots$ of approximations to $\sqrt{2}$ in such a way that the remainder terms

$$s_n = x_n - \sqrt{2}$$

[1] *Hint:* $m_5 = 45/32$.

[2] An integer k is *even* if $k/2$ is an integer, and odd otherwise. We are using here the fact that the square of an even integer is even and the square of an odd integer is odd.

[3] *Hint:* Write $n' = n(\frac{m}{n} - k)$.

diminish rapidly. Squaring both sides yields $s_n^2 = x_n^2 - 2\sqrt{2}x_n + 2$, and solving for $\sqrt{2}$ yields

$$\sqrt{2} = \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big) - \frac{s_n^2}{2x_n}.$$

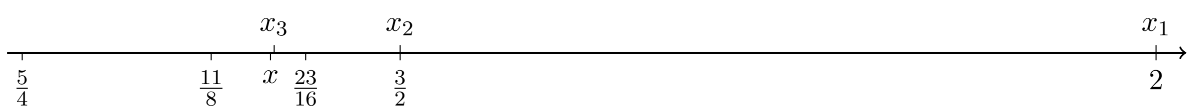
We accordingly define

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big), \tag{4.1}$$

because x_{n+1} is a better approximation than x_n as long as $|s_n| < 2x_n$. Note that the summands x_n and $\frac{2}{x_n}$ are equal if $x_n^2 = 2$, and in that case the next term x_{n+1} equals the previous x_n. Otherwise the summands are different and x_{n+1} is obtained by averaging them.

A simple choice of starting value is $x_1 = 2$, yielding $x_2 = \frac{1}{2}(2 + \frac{2}{2}) = \frac{3}{2}$, $x_3 = \frac{1}{2}(\frac{3}{2} + \frac{2}{3/2}) = \frac{17}{12}$, and so on. We obtain a number line and table, analogous to but better than the ones of Section 2.

n	1	2	3
x_n	2	$\frac{3}{2}$	$\frac{17}{12}$
x_n^2	$4 = 2 + 2$	$\frac{9}{4} = 2 + \frac{1}{4}$	$\frac{289}{144} = 2 + \frac{1}{144}$



It is clear that the values $x_1, x_2, \ldots$ approach x much more rapidly than the $m_1, m_2, \ldots$ of Section 2.

Exercise 4.2. (1) Add a column to the table above for $n = 4$, and a row with M_n defined by the equation $x_n^2 = 2 + \frac{1}{M_n}$. Thus $M_1 = \frac{1}{2}$, $M_2 = 4$, and so on. These values of M_n measure the accuracy of the successive terms in the sequence. Admire how quickly they grow.[4]

(2) Suppose $x_n^2 > 2$, and let $M_n = 1/(x_n^2 - 2)$. Find positive integers a and b such that if $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$, then[5] $M_{n+1} = aM_n^2 + bM_n$.

The best way to measure the accuracy of the approximation is in terms of the relative remainders

$$r_n = \frac{x_n - \sqrt{2}}{\sqrt{2}}. \tag{4.3}$$

Putting together (4.3) and (4.1), combining terms, and factoring the numerator, yields

$$r_{n+1} = \frac{1}{2\sqrt{2}}\Big(x_n + \frac{2}{x_n}\Big) - 1 = \frac{x_n^2 - 2\sqrt{2}x_n + 2}{2x_n\sqrt{2}} = \frac{(x_n - \sqrt{2})^2}{2x_n\sqrt{2}} = \frac{r_n^2}{x_n\sqrt{2}} = \frac{r_n^2}{2r_n + 2}.$$

Using $x_2 = 3/2$ and $1.4 < \sqrt{2} < 1.5$ in (4.3) gives

$$0 < r_2 < .1/\sqrt{2}.$$

[4] *Hint:* $M_4 = 166464$.

[5] *Hint:* Substitute $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$ into $M_{n+1} = 1/(x_{n+1}^2 - 2)$ and simplify to get $4x_n^2/(x_n^2 - 2)^2$. Then plug in $x_n^2 = 2 + \frac{1}{M_n}$.

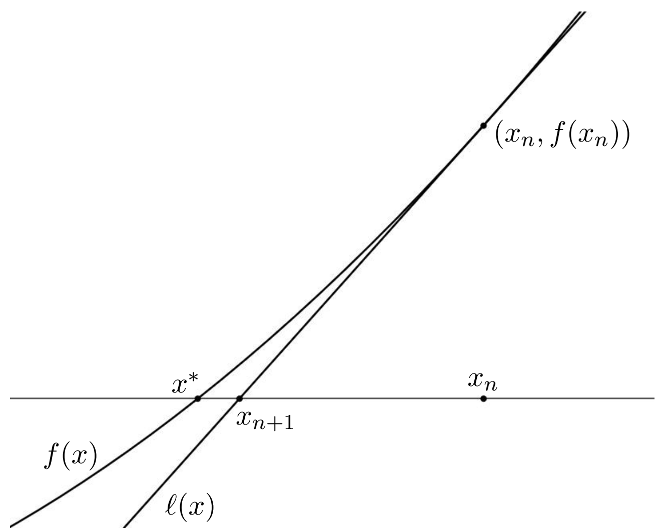
Next, since since $0 < r_{n+1} < r_n^2/2$, we get $0 < r_3 < 2/800$, $0 < r_4 < 2/800^2$, and more generally

$$0 < r_{n+3} < 2/800^{2^n}, \text{for } n = 0, 1, 2, \dots \tag{4.4}$$

EXERCISE 4.5. (1) For a square the size of North America, find n such that x_n computes the diagonal to accuracy better than the width of a hair.[6]

(2) Use the above method to derive a sequence of approximations to $\sqrt{3}$. Start with $x_1 = 2$, find formulas for x_{n+1} in terms of x_n and r_{n+1} in terms of r_n, and use $1.73 < \sqrt{3} < 1.75$ to show that $0 < r_2 < .02/\sqrt{3}$ and deduce an even more spectacular analogue of (4.4). This solves the problem of finding the diagonal of a cube in terms of its sides.

5. **Coming attractions.** Later we will see that the method of recursive averages above is a special case of *Newton's method* of solving equations. Specifically, given an approximate solution x_n to $f(x) = x^2 - 2 = 0$, we find an improved solution x_{n+1} as follows: let $\ell(x)$ be the tangent line at x_n to $f(x)$, and let x_{n+1} be the solution to the linear equation $\ell(x_{n_1}) = 0$:



This works especially well for extracting kth roots, yielding the recursion

$$x_{n+1} = \frac{1}{k}\Big((k-1)x_n + \frac{q}{x_n^{k-1}}\Big)$$

for computing $\sqrt[k]{q}$.

For more difficult problems we will develop the theory of integration. The classic in this genre is the computation of π, the area of the unit disk. Archimedes' method of exhaustion handles this by computing areas of inscribed and circumscribed polygons, and only with great effort and ingenuity was he able to find in this way that $3\frac{1}{7} < \pi < 3\frac{10}{71}$.

We can do better by paralleling our approach to $\sqrt{2}$, using a trigonometric function such as $f(x) = \sin(x)$, which obeys $f(\pi/6) = 1/2$. The difficulty is evaluating f accurately. This can be handled by

[6]*Hint:* If the diagonal of the square is 10^7 meters, then the square is bigger than North America. But $r_5 < 2/800^4 = 2^{-11}10^{-8}$, so the accuracy attained by $x_5 = \frac{665857}{470832}$ is better than $2^{-11}10^{-1} < 5 \cdot 10^{-5}$ meters, which is less than the width of a hair. For a square the size of the Solar System, x_6 is more than accurate enough.

writing

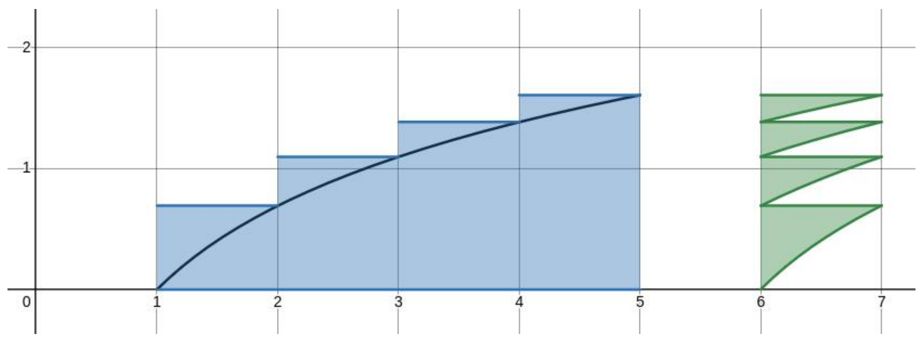
$$\pi = 6 \arcsin\left(\frac{1}{2}\right) = 6\Big(\frac{1}{2} + \frac{1}{6}\Big(\frac{1}{2}\Big)^3 + \frac{3}{40}\Big(\frac{1}{2}\Big)^5 + \cdots\Big)$$

(or better yet, a strategic variant of this equation), where the series expansion of arccos is obtained by integration.

We will see that integration solves many problems. For example, $n! = 1 \cdot 2 \cdots n$ becomes difficult to compute directly as n grows. But by taking the logarithm we can relate it to an easy integral. We write

$$\ln(n!) = \ln 1 + \ln 2 + \cdots \ln n,$$

and observe that this is a sum of rectangles sitting above the curve $\ln x$, pictured below with $n = 5$:



Hence, if $n \geq 2$ then

$$\int_1^n \ln x \, dx < \ln n!$$

On the other hand, we will see that these rectangles exceed the area under the curve by an amount (the area of the green curved triangles above) less than $\frac{1}{2} \ln n$, which yields

$$\int_1^n \ln x \, dx < \ln n! < \int_1^n \ln x \, dx + \frac{1}{2} \ln n.$$

Evaluating the integrals gives

$$n \ln n - n + 1 < \ln n! < n \ln n - n + \frac{1}{2} \ln n + 1,$$

and exponentiating gives

$$e\Big(\frac{n}{e}\Big)^n < n! < e\sqrt{n}\Big(\frac{n}{e}\Big)^n,$$

The right inequality is much closer than the left one, and with more work one can prove the more precise *Stirling's approximation*, which states that

$$\sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n < n! < \sqrt{2\pi n} e^{\frac{1}{12n}} \Big(\frac{n}{e}\Big)^n.$$

A more complicated problem we will solve using integration and iteration is the differential equation

$$\frac{d}{dt} x(t) = f(x(t), t), \qquad x(0) = x_0,$$

where x_0 is a given initial condition and f is a given function for the law obeyed by y. This is solved by the iteration

$$x_{n+1}(t) = y_0 + \int_0^t f(x_n(s), s) \, ds.$$

For this problem we iterate over functions $x_n(t)$, where previously we iterated over numbers x_n.

References

[Dat] Radoslav Datchev, *An investigation of knowledge*.
[Ded] Richard Dedekind, *Continuity and irrational numbers*, 1872.
[HaWr] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 1938.
[New] Sir Isaac Newton, *Analysis by equations of an infinite number of terms*, 1669.