Sequences and series of functions

1. **Introduction.** We will show that, if α is real and $|x| < 1$, then

$$(1+x)^\alpha = f(x), \qquad \text{where } f(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots . \tag{1.1}$$

If α is a positive integer then then the series terminates and this is true for all x; it is the binomial expansion. The more general version is valuable for analyzing expressions like the Lorentz factor in relativity:

$$\gamma = 1/\sqrt{1-\tfrac{v^2}{c^2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots ,$$

which combines with the energy formula $E = \gamma mc^2$ to yield

$$E = mc^2 + \frac{1}{2}mv^2 + \cdots .$$

The first term, mc^2, is independent of motion and can be ignored when computing dynamics. The second term $\frac{1}{2}mv^2$ is the nonrelativistic kinetic energy and is the only one which needs to be taken into account, unless v is a significant fraction of the speed of light c.

To prove (1.1), we observe that, since $\frac{d}{dx}[(1+x)^\alpha] = \alpha(1+x)^{\alpha-1}$, we have

$$(1+x)\frac{d}{dx}\Big[(1+x)^\alpha\Big] = \alpha(1+x)^\alpha.$$

Suppose we can show similarly that

$$(1+x)f'(x) = \alpha f(x). \tag{1.2}$$

Then it will follow that

$$\frac{d}{dx}\Big[(1+x)^{-\alpha}f(x)\Big] = 0,$$

which implies (1.1).

Proving (1.2) requires differentiating (1.1) term by term to obtain

$$f'(x) = \alpha + \alpha(\alpha-1)x + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}x^2 + \cdots ,$$

and we will now develop the technique to see that this is correct.

The difficulty is that going from (1.1) to (1.2) requires writing

$$f'(x) = \frac{d}{dx}\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty}\frac{d}{dx}s_n(x),$$

and switching the $\frac{d}{dx}$ and $\lim_{n\to\infty}$ is not always correct.

Example 1.3. Let

$$f_n(x) = \frac{x}{1+n^2x^2},$$

then

$$\lim_{n\to\infty} f_n(x) = 0, \qquad \text{for all } x,$$

Kiril Datchev, April 14, 2026.

but

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-n^2x^2}{(1+n^2x^2)^2} = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Thus

$$\frac{d}{dx}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x), \qquad \text{only when } x \neq 0.$$

See Figure 1

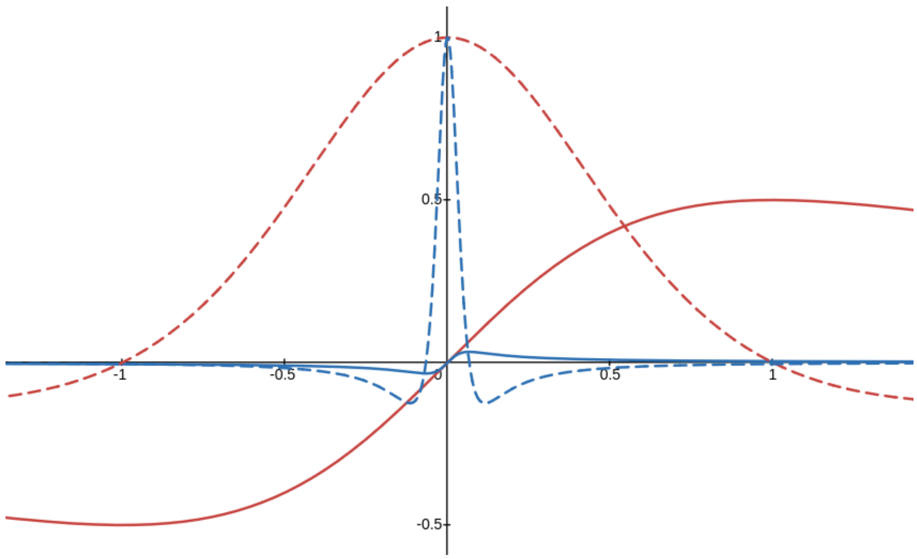

FIGURE 1. The functions from Example 1.3, with f_1 in solid red, f_1' in dashed red, f_{15} in solid blue, and f_{15}' in dashed blue. The functions f_n are tending to zero, but in such a way that the derivatives at 0 stay equal to 1, which breaks the equation $\frac{d}{dx}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x)$ at $x = 0$.

Similar issues arise with integration:

EXERCISE 1.4. Let p be a real number, and let $f_n(x) = n^p(x^n - x^{n+1})$.

(1) Find $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$ in terms of p.
(2) For which values of p do we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx?$$

We will see that power series, however, are generally quite safe. To obtain a precise result, and also to get good more general conditions under which these switches are valid, we need a new concept.

2. **Uniform convergence.** We say that $f_n(x)$ converges to $f(x)$ *uniformly* for all x in E, when, for any $\varepsilon > 0$, there is N such that $n > N$ implies $|f_n(x) - f(x)| < \varepsilon$ for all x in E. The key point is that the same N must work for all x in E; this is called finding a uniform N.

EXAMPLE 2.1. Let $f_n(x) = x^n$. Then $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0, 1)$. See Figure 2.

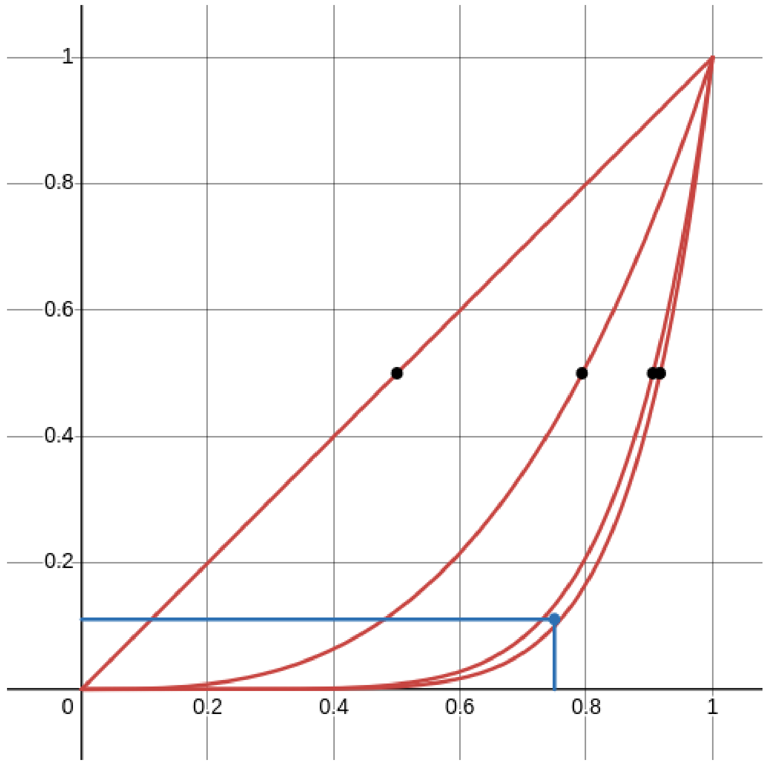

FIGURE 2. In red, the graphs of $f_n(x)$ from Example 2.1, for $n = 1, 3, 7, 8$. The blue is the line $y = \varepsilon = 1/9$ and the line $x = b = 3/4$. We see that, with these values of ε and b, if $n > 7$, then $|f_n(x) - f(x)| < \varepsilon$ for $x \in (0, b)$. The black dots are the corresponding points $(1/2, (1/2)^{1/n}$, which are used to show that the convergence is not uniform on $(0, 1)$.

(1) If $0 < b < 1$, then the convergence is uniform on $(0, b)$. To check this, calculate

$$|f_n(x) - f(x)| = x^n < \varepsilon \qquad \iff \qquad n > \frac{\ln \varepsilon}{\ln x}.$$

Then we may take $N = \frac{\ln \varepsilon}{\ln b}$, so that $n > N$ implies $n > \ln \varepsilon / \ln x$, using $\ln x < \ln b$.

(2) The convergence is not uniform on $(0, 1)$. To prove it, observe that $f_n((1/2)^{1/n}) = 1/2$, which implies that there is no N that will make $f_n(x) < 1/2$ for all $n > N$ and for all $x \in (0, 1)$.

The statement that $f_n(x)$ converges to $f(x)$ uniformly for all x in E is equivalent to the statement that there is a sequence of real numbers $B_n \to 0$ such that $|f_n(x) - f(x)| \leq B_n$ for all x in E, and for n large enough. The statement that $f_n(x)$ does not converge to $f(x)$ uniformly for all x in E is equivalent to the statement that there exist $\varepsilon > 0$ and a sequence $x_1, x_2, \ldots$ in E such that $|f_n(x_n) - f(x_n)| \geq \varepsilon$ for infinitely many n.

EXERCISE 2.2. For each of the following functions, determine the pointwise limit $f(x)$ on the indicated interval, and determine whether the convergence is uniform. If the convergence is uniform, find a sequence of real numbers $B_n \to 0$ such that $|f_n(x) - f(x)| \leq B_n$ for all x in the interval. If it isn't, find $\varepsilon > 0$ and a sequence x_n in the interval such that $|f_n(x_n) - f(x_n)| \geq \varepsilon$ for all n.

(1) $f_n(x) = x^{1/n}$ on $[0, 1]$.

(2) $f_n(x) = \frac{1-n^2x^2}{(1+n^2x^2)^2}$ on $(10^{-10}, \infty)$.
(3) $f_n(x) = \frac{1-n^2x^2}{(1+n^2x^2)^2}$ on $(0, \infty)$.

Uniform convergence can be used to justify equations of the form

$$\lim \int = \int \lim, \qquad \frac{d}{dx}\lim = \lim \frac{d}{dx}, \qquad \lim_n \lim_x = \lim_x \lim_n . \tag{2.3}$$

We will treat these in order of difficulty, beginning with the simplest.

THEOREM 2.4. *If $f_n \to f$ uniformly on $[a, b]$, and all the functions $f, f_1, f_2, \ldots$ are integrable, then*

$$\int_a^b f_n \to \int_a^b f.$$

Proof. Since $f_n \to f$ uniformly on $[a, b]$, there is a sequence $B_n \to 0$ such that $|f_n(x) - f(x)| \le B_n$. Hence

$$\Big| \int_a^b f_n - \int_a^b f \Big| \le \int_a^b |f_n - f| \le \int_a^b B_n = (b-a)B_n \to 0.$$

□

THEOREM 2.5. *If I is an open interval, and $f_n \to f$ uniformly on I, and $f_n' \to g$ uniformly on I, and all the functions are continuous, then $g = f'$.*

Proof. We use the fundamental theorem of calculus to get this from Theorem 2.4. Let a and b be points in I. We will show that

$$\int_a^b g = f(b) - f(a). \tag{2.6}$$

This implies the desired result upon differentiating both sides with respect to b. To prove (2.6), start with

$$\int_a^b f_n' = f_n(b) - f_n(a)$$

and take the limit of both sides as $n \to \infty$, using Theorem 2.4 to obtain $\int_a^b f_n' \to \int_a^b g$. □

THEOREM 2.7. *If $f_n \to f$ uniformly on E, and if each f_n is continuous, then f is continuous.*

Proof. Let $p \in E$ and $\varepsilon > 0$ be given. Take $B_n \to 0$ such that $|f(x) - f_n(x)| \le B_n$ for all x in E and for all n large enough. Then

$$\begin{aligned} |f(x) - f(p)| &= |f(x) - f_n(x) + f_n(x) - f_n(p) + f_n(p) - f(p)| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(p)| + |f_n(p) - f(p)| \\ &\le B_n + |f_n(x) - f_n(p)| + B_n \end{aligned}$$

First take n such that $B_n < \varepsilon/3$, and then take δ such that $|x - p| < \delta$ implies $|f_n(x) - f_n(p)| < \varepsilon/3$. Then $|x - p| < \delta$ implies

$$|f(x) - f(p)| < \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon.$$

□

The proof of Theorem 2.7 can be varied to show that $\lim_x \lim_n = \lim_n \lim_x$ for various notions of limit, as long as convergence is uniform. Here is an example:

Exercise 2.8. Let $f_n \to f$ uniformly on $(0, \infty)$. Let L_1, L_2, ... be a sequence of real numbers converging to a real number L. Prove that if $\lim_{x\to\infty} f_n(x) = L_n$, then $\lim_{x\to\infty} f(x) = L$.

Now we establish the simplest and most important test of uniform convergence.

Theorem 2.9 (Weierstrass M test). *Let $\sum_k M_k$ be a convergent series. If $|g_k(x)| \le M_k$ for all x in E and for all k, then $\sum_k g_k(x)$ converges uniformly.*

Proof. We know that $\sum_k g_k(x)$ converges by the comparison test. To check that the convergence is uniform, let $f_n(x) = \sum_{k=1}^n g_k(x)$, $f(x) = \sum_{k=1}^\infty g_k(x)$, and write

$$\begin{aligned} |f_n(x) - f(x)| &= \Big|\sum_{k=1}^n g_k(x) - \sum_{k=1}^\infty g_k(x)\Big| = \Big|\sum_{k=n+1}^\infty g_k(x)\Big| \le \sum_{k=n+1}^\infty |g_k(x)| \\ &\le \sum_{k=n+1}^\infty M_k = \sum_{k=1}^\infty M_k - \sum_{k=1}^n M_k \to \sum_{k=1}^\infty M_k - \sum_{k=1}^\infty M_k = 0. \end{aligned}$$

Thus we can use $B_n = \sum_{k=n+1}^\infty M_k$. □

Exercise 2.10. Prove that $f(x) = \sum_{k=2}^\infty \frac{x^{k-1}}{2^k(1+x^k)}$ is continuous for $x \ge 0$, and evaluate[1] $\int_0^1 f(x)\,dx$.

The M test leads to a clean theory of power series. The main result is this:

Theorem 2.11. *Suppose $\sum_{k=0}^\infty a_k p^k$ converges, and $0 < b < p$. Then $\sum_{k=0}^\infty a_k x^k$ converges uniformly on $[-b, b]$.*

Proof. Since $a_k p^k \to 0$ as $k\infty$, the sequence is bounded, i.e. there is C such that $|a_k p^k| \le C$ for all k. Hence, if $x \in [-b, b]$, then

$$|a_k x^k| \le C p^{-k} |x|^k \le C p^{-k} b^k,$$

so we can use $M_k = C p^{-k} b^k$, thanks to the fact that $p^{-1} b \in (0, 1)$. □

Example 2.12. Let

$$f(x) = \frac{x^2}{2\cdot 1} + \frac{x^3}{3\cdot 2} + \frac{x^4}{4\cdot 3} + \cdots \tag{2.13}$$

Then we expect

$$f'(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \tag{2.14}$$

and

$$f''(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}. \tag{2.15}$$

Since the last series converges when $|x| < 1$, it follows that it converges uniformly on $[-b, b]$ for any $b < 1$, and hence it follows by Theorem 2.4 that the equations (2.13), (2.14), (2.15) are correct for any $x \in (-1, 1)$. Integrating and using $f(0) = f'(0) = 0$ yields

$$f'(x) = -\ln(1-x), \qquad f(x) = x + (1-x)\ln(1-x), \qquad \text{for } x \in (-1, 1).$$

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[1] *Hint:* For the proof of continuity, bound the fraction in a different way depending on whether $x \le 1$ or $x \ge 1$. For the evaluation of the interval, use one of the series for $\ln 2$ from the chapter on Approximation and Asymptotics.

Note that $f(1/2)$ gives a series for $\ln(1/2) = -\ln 2$ which converges faster than the series given by $f'(1/2)$, which converges faster than

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ,$$

which corresponds to $f'(-1)$, and which confirms that our formula for f' is correct at the endpoint $x = -1$, even though the tests used to obtain it are not valid there. Similarly, by partial fraction decomposition,

$$\frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1 = f(1),$$

which confirms that our formula for f is correct at the endpoint $x = 1$.[2]

Example 2.16. Beautiful examples are afforded by Fourier series. Consider the three functions

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}, \qquad g(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}, \qquad h(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{-k}. \tag{2.17}$$

We would like to have

$$f' = g, \qquad g' = h, \tag{2.18}$$

and, properly interpreted, this is correct. For $f' = g$, there is no difficulty with interpretation, and the proof is as in Example 2.12, applying the M test with $M_k = 1/k^2$. The result is confirmed by Figure 3.

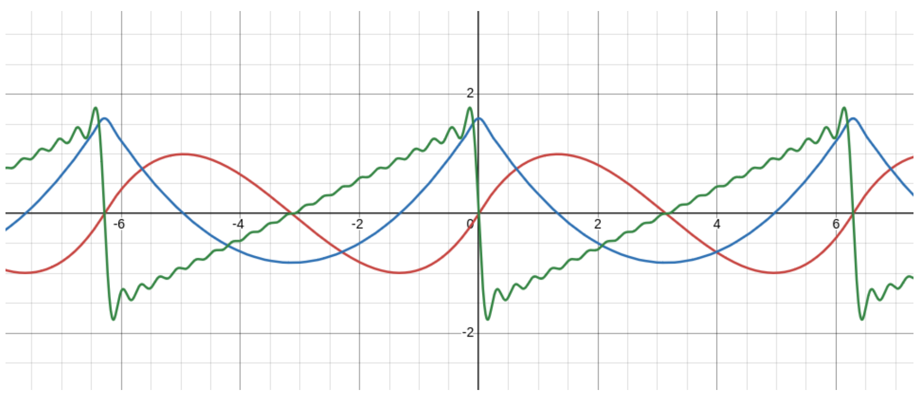

Figure 3. In red, blue, and green, the graphs of f, g, and h from (2.17).

But Figure 3 also supports a version of $g' = h$, even though something is clearly going wrong near $x = 2m\pi$. A complete understanding of (2.18) requires serious Fourier analysis; see Seeley's *Introduction to Fourier Series and Integrals* for a beautiful presentation of the basics. Here we just emphasize that the tests we are proving provide sufficient but not necessary conditions for interchange of operations as in (2.3).

[2] It is sometimes of value to go even beyond the endpoint, into the realm of equations like

$$1 + 2 + 4 + 8 + \cdots = -1;$$

in this case we say that convergence holds with respect to the 2-adic metric, which is important in number theory, and can be used to solve Diophantine equations (see the first lecture here `https://sites.math.rutgers.edu/~alexk/2023S572/index.html`). For an example of such a sum in physics, see page 26 of `https://arxiv.org/pdf/quant-ph/0106045.pdf`. The most accessible example of a physical meaning of a divergent series is Madelung's constant, which is the 3 dimensional version of the crystal energy computation from the chapter on Sequences.

It is also interesting to note that the graph of $h(x)$ looks linear on $(0, 2\pi)$. Since we know $h(\pi) = 0$ (because all terms in the sum are zero), while

$$h\left(\frac{\pi}{2}\right) = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots = -\frac{\pi}{4}.$$

This suggests

$$h(x) = \frac{1}{2}x - \frac{\pi}{4},$$

which is true, although the proof would take us some work at this point (see Theorem 1-4 of Seeley's book mentioned above). Taking it as given, we expect

$$g(x) = \frac{1}{4}x^2 - \frac{\pi}{2}x + C, \tag{2.19}$$

where C is a constant of integration. To determine C, take (2.19) $\int_0^\pi dx$ of (2.19) to get

$$0 = \frac{1}{12}\pi^3 - \frac{\pi}{4}\pi^2 + C\pi,$$

or $C = \pi^2/6$. Plugging in $x = 0$ yields

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{2.20}$$

the famous Basel sum. There are many ways to evaluate the Basel sum, but all are tricky and indirect. The quickest way to make the above computation complete is to prove that

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{1}{4}x^2 - \frac{\pi}{2}x + \frac{\pi^2}{6}$$

using[3] Chapter 2 and the first page of Chapter 9 of Körner's *Fourier Analysis*, which is a fun book made up of a large number of short chapters that can be read in almost any order.

Exercise 2.21. In each of these problems, carefully explain the role of uniform convergence at each step.

(1) Evaluate[4]

$$\frac{2}{3^3} + \frac{3}{3^4} + \frac{4}{3^5} + \frac{5}{3^6} + \cdots.$$

(2) Let $f_m(x) = x^{23}(1 - x^m)^{-3}$. For each positive integer m, find a power series[5] for f_m and find the thirtieth derivative of f_m at $x = 0$.

(3) Consider the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Prove that ζ, ζ', ζ'', ζ''', etc. are all defined and continuous for $s > 1$.

[3] But see also the Corollary to Theorem 1-2 of Seeley's book mentioned above.

[4] *Hint:* It is helpful to sum $2x + 3x^2 + 4x^3 + \cdots$.

[5] *Hint:* It is helpful to find the series of $(1 - u)^{-3}$ by differentiating the geoemtirc series.