Sequences

We now turn to more general problems regarding sequences $s_1, s_2, \dots$.

1. **Mathematical induction.** In the Introduction we defined a sequence of approximations to $\sqrt{2}$ by

$$x_1 = 2, \qquad x_{n+1} = \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big),$$

and defined the relative errors

$$r_n = \frac{x_n - \sqrt{2}}{\sqrt{2}}.$$

We obtained the recursive formula

$$r_{n+1} = \frac{r_n^2}{2r_n + 2}, \qquad n \geq 1,$$

which shows that

$$n \geq 1 \text{ and } r_n > 0 \Longrightarrow r_{n+1} > 0. \tag{1.1}$$

The symbol '$\Longrightarrow$' means 'implies'. On the other hand, since $r_1 = \frac{2-\sqrt{2}}{\sqrt{2}}$, we have

$$r_1 > 0. \tag{1.2}$$

Combining (1.1) and (1.2) yields

$$r_n > 0 \text{ for all } n \geq 1.$$

More generally, combining

$$n \geq n_0 \text{ and the property } P \text{ is true of } s_n \Longrightarrow \text{ the property } P \text{ is true of } s_{n+1}, \tag{1.3}$$

and

$$\text{the property } P \text{ is true of } s_{n_0}, \tag{1.4}$$

yields

$$\text{the property } P \text{ is true of } s_n \text{ for all } n \geq n_0.$$

This is the *principle of mathematical induction*; (1.3) is the *inductive step*, and (1.4) is the *base step*. We visualize this as a line of dominos falling: the inductive step is having the dominos set up such that each knocks over the next when it falls, and the base step is knocking over the first domino.

In the example above, we used the principle with $n_0 = 1$, P the property of being > 0, and $s_n = r_n$.

Example 1.5. Let us use mathematical induction to show that

$$r_n < 2/800^{2^{n-3}} \text{ for all } n \geq 3. \tag{1.6}$$

To establish the base step, we use $x_2 = 1.5$ and $0 < 1.5 - \sqrt{2} < 0.1$ to deduce that

$$0 < r_2 < \frac{1}{10\sqrt{2}},$$

and hence

$$r_3 = \frac{r_2^2}{2r_2 + 2} < \frac{r_2^2}{2} < \frac{(1/10\sqrt{2})^2}{2} = \frac{2}{800}.$$

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To establish the inductive step, we observe that if

$$r_n < 2/800^{2^{n-3}},$$

then

$$r_{n+1} < \frac{r_n^2}{2} < \frac{(2/800^{2^{n-3}})^2}{2} = \frac{2}{(800^{2^{n-3}})^2} = \frac{2}{800^{2^{n-2}}}.$$

Exercise 1.7.

(1) Use mathematical induction to prove that if $s_1 = 1$ and $s_{n+1} = s_n/2$ for all $n \geq 1$, then $s_n = 1/2^n$ for all $n \geq 1$.

(2) Define a sequence recursively by $s_1 = 1/5$ and $s_{n+1} = s_n^3/9$ for all $n \geq 1$. Mimic the treatment of r_n above to find a formula for s_n in the style of (1.6), and prove it by mathematical induction.

2. **Long term behavior.** We analyze the long term behavior of a sequence using a key concept which has several equivalent formulations:

(1) A property P is is true of s_n for all but finitely many n.

(2) A property P is is true of s_n for n large enough.

(3) There exists a number N such that, if $n \geq N$, then P is true of s_n.

(4) There exists a number N such that, if $n > N$, then P is true of s_n.

We sometimes express this concept by saying that P is true of s_n *eventually.*

Example 2.1. Let $s_n = n^2 - 3n - 108$. Then $s_n > 0$ for all but finitely many n, as we can see in three ways, the first rough and the last two precise.

(1) Sketch the parabola $x^2 - 3x - 108$ as V and note that when x is large, this graph must be above the x axis, regardless of where the x axis is.

(2) To find an explicit N, we pull out the dominant term, and write

$$s_n = n^2\Big(1 - \frac{3}{n} - \frac{108}{n^2}\Big).$$

Now use the fact that the right side is positive as long as both factors are positive, so it is enough to take $N = 20$: then, for $n \geq N$, we have

$$1 - \frac{3}{n} - \frac{108}{n^2} > 1 - \frac{3}{20} - \frac{1}{3} > 0.$$

(3) We can also find the best N by factoring $n^2 - 3n - 108 = (n - 12)(n + 9)$ to see that $s_n \leq 0$ when $n \leq 12$ and $s_n > 0$ when $n \geq 13$.

Example 2.2. If $s_n = n^2 + 36\sqrt{2}(-1)^n n + 10^5 \cos(\frac{\pi}{20} n)$, then $s_n > 100$ for all but finitely many n. This statement is more complicated but pulling out the dominant term still works well:

$$s_n - 100 = n^2\Big(1 - \frac{36\sqrt{2}(-1)^n}{n} + \frac{10^5 \cos(\frac{\pi}{20} n) - 100}{n^2}\Big).$$

Now we may take $N = 1000$ and, for $n \geq N$, deduce

$$1 - \frac{36\sqrt{2}(-1)^n}{n} + \frac{10^5 \cos(\frac{\pi}{20} n) - 100}{n^2} > 0,$$

from

$$\frac{36\sqrt{2}(-1)^n}{n} < \frac{48}{1000} \text{ and } \frac{10^5\cos(\frac{\pi}{20}n) - 100}{n^2} > -\frac{1}{10} - \frac{1}{10000}.$$

Finding the best N would be a bigger job, but we can quickly see that we are within a factor of 10 by writing

$$s_{100} = 1 - \frac{36\sqrt{2}(-1)^{100}}{100} + \frac{10^5\cos(\frac{\pi}{20}100) - 100}{100^2} = 1 - \frac{36\sqrt{2}}{100} - 10 - \frac{1}{100} < 0.$$

Exercise 2.3.

(1) Let $s_n = n^2 + 4n - 100$. Find N such that $a_n > 1000$ when $n \geq N$.

(2) Let $s_n = -n^4 + 37n + (-1)^n 10^9$. Find N which is within a factor of 10 of the best N such that $s_n < 0$ when $n \geq N$.

3. **Limits.** We say that a sequence s_1, s_2, ... *converges* to a *limit* L when, for any $\varepsilon > 0$, we have $|s_n - L| < \varepsilon$ for all but finitely many n. We write

$$\lim_{n\to\infty} s_n = L, \qquad \text{or} \qquad \lim s_n = L.$$

Graphically, the meaning of this definition is the following:

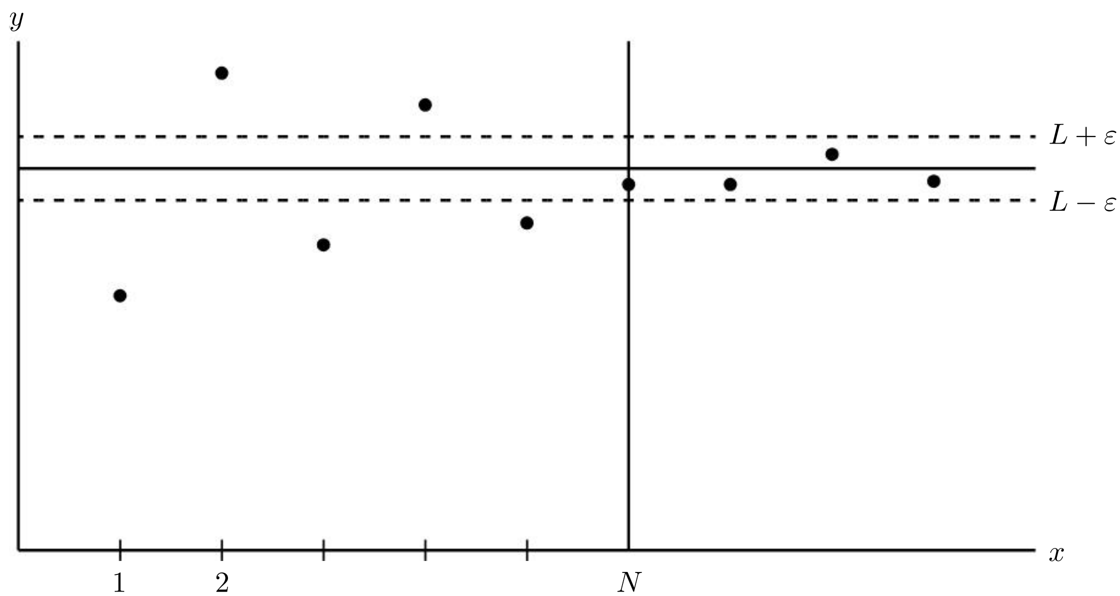

Figure 1. The limit of a sequence. The dots are the graph of the terms of the sequence: $(1, s_1)$, $(2, s_2)$, and so on. Given any tolerance $\varepsilon > 0$, there is N such that all terms of the sequence from s_N onward are within that tolerance of the limit: $|s_n - L| < \varepsilon$, i.e. all dots to the right of the line $x = N$ are between the lines $y = L + \varepsilon$ and $y = L - \varepsilon$.

Analogously to the discussion of long term behavior above, it makes difference to the definition if we replace the condition $|s_n - L| < \varepsilon$ by $|s_n - L| \leq \varepsilon$.

The most direct way to prove that a sequence s_1, s_2, ... converges to a limit is to let $\varepsilon > 0$ be given, and find a corresponding N such that $n \geq N$ implies $|s_n - L| < \varepsilon$.

Example 3.1. Let $s_n = 1/n$. Given $\varepsilon > 0$, we have $0 < s_n < \varepsilon$ when $n > 1/\varepsilon$. Hence $\lim s_n = 0$.

Example 3.2. Consider the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$. Then $s_n = \frac{n-1}{n} = 1 - \frac{1}{n}$. As above, $-\varepsilon < s_n - 1 < 0$ when $n > 1/\varepsilon$. Hence $\lim s_n = 1$.