MA 362 first midterm review problems

- The first midterm will be on Monday, February 6th, from 8:00 to 9:30 pm, in CL50 224.
- It will cover all the material from the classes and homework up to that date.
- Most of the problems on the exam will be closely based on ones from the list below (but the actual exam will be much shorter).
- For each problem, you must explain your reasoning.
- Note that these are not arranged in order of difficulty!

1. Let $(a, b, 0)$, $(0, a, b)$, and $(a, 0, b)$ be three (not necessarily distinct) points in $\mathbb{R}^3$, where a and b are real numbers.
   (a) For which real values of a and b are the points collinear?
   (b) For which values are they coplanar with the origin?
   (c) How do the answers change if the third point is replaced by $(b, 0, a)$?

2. (a) Find a normal unit vector to the plane in $\mathbb{R}^3$ given by $x-2y-2z = 1$.
   (b) Find the intersection of this plane with the line perpendicular to it and passing through $(1, 2, 3)$.
   (c) What is the distance between the plane and the point $(1, 2, 3)$?

3. What is the distance between the two planes given by the equations $x + 2y + 3z = 0$ and $x + 2y + 3z = 4$?

4. What is the cosine of the angle between the diagonal of a cube and one of the edges? What about between the diagonal of a cube and the diagonal of one of its sides?

5. What are the smallest possible numbers a and b such that
$$\|x + y\|\|x - y\| \leq a\|x\|^2 + b\|y\|^2,$$
for any x and y in $\mathbb{R}^n$?

6. (a) Find the equation of the sphere in $\mathbb{R}^3$ with center $(1, 1, 1)$ and passing through $(2, 2, 2)$.
   (b) What is the intersection of this sphere with the xy-plane?

7. Find the equation for the surface consisting of those points in $\mathbb{R}^3$ which

are equidistant from the y-axis and the xz-plane. Sketch and describe the surface.

8. Let n be an integer. For which values of n does the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^{2n} + y^2}$$

exist?

9. Let

$$f(x,y) = \begin{cases} x, & \text{if } |y| > |x|, \\ -x, & \text{if } |y| \le |x|. \end{cases}$$

(a) At which points (x_0, y_0) do the partial derivatives of f exist? What are the values of the partial derivatives at those points?

(b) At which points is f continuous?

(c) At which points is f differentiable?

10. Let

$$f(x,y) = \begin{cases} \dfrac{(x^2 - y^4)^2}{(x^2 + y^4)^2}, & \text{if } (x,y) \neq (0,0), \\ 1, & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Show that the directional derivatives of f exist at $(0,0)$ in all directions. What are their values?

(b) Is f continuous at $(0,0)$?

(c) Is f differentiable at $(0,0)$?

11. Let $f = f(u,v,w)$ be a differentiable function $\mathbb{R}^3 \to \mathbb{R}$, and let

$$g(x,y) = f(x-y, x+y, xy)$$

(a) If $(4, 10, 21)$ is a critical point of f, which point must be a critical point of g?

(b) Calculate $\partial_x \partial_y g$ in terms of the derivatives of f.

12. A train track is to be built up a hill given by the graph in $\mathbb{R}^3$ of the function $f(x,y) = 10 - x^2 - 2y^2$. What are the possible directions the

track can head starting at $(1,0)$ so that it is going uphill at a slope of $1/2$?

13. Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is differentiable and satisfies

$$\nabla f(1,-e,e) = (1,2,3),$$

and that $z = z(x,y)$ is given by the implicit equation

$$x - y + z + \log z = 2 + 2e,$$

where x and y are independent variables. If $g(x,y) = f(x,y,z(x,y))$, what is $\nabla g(1,-e)$?

Formula sheet

- Cylindrical coordinates in $\mathbb{R}^3$ are given by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Spherical coordinates are given by $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$.