Differential forms in two dimensions

Kiril Datchev, March 23, 2017

1 Three integral formulas

Below we will use the concept of a differential form to explain the relationship between the following three integral formulas. Along the way we develop the algebra and calculus of these differential forms, and highlight the correspondences between properties of these and more familiar facts from vector algebra and calculus.

We stick to the two dimensional case here. Later we will consider the three dimensional case in some detail also, and see how the patterns revealed by the language of differential forms extend into higher dimensions as well.

Throughout we assume for simplicity that all functions are differentiable infinitely many times.

1.1 Fundamental Theorem of Calculus for line integrals

The line integral version of the Fundamental Theorem of Calculus says that if $f = f(x, y)$ then

$$\int_C \partial_x f dx + \partial_y f dy = f(q) - f(p), \tag{1}$$

where C is an oriented curve in $\mathbb{R}^2$ from p to q.

1.2 Green's Theorem

Green's Theorem says that if $F_1 = F_1(x, y)$ and $F_2 = F_2(x, y)$ then

$$\iint_D (\partial_x F_2 - \partial_y F_1) dA = \int_{\partial D} F_1 dx + F_2 dy, \tag{2}$$

where D is a simple region (or a union of simple regions) and ∂D is the boundary of D oriented such that the interior of D is to the left of ∂D.

1.3 Change of variables for double integrals

The change of variables formula for double integrals says that if $x = x(u, v)$ and $y = y(u, v)$ is a change of variables, (so that we can write $u = u(x, y)$ and $v = v(x, y)$) and $f = f(x, y)$, then

$$\iint_D f dxdy = \iint_{D^*} f \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv, \tag{3}$$

where $\partial(x, y)/\partial(u, v) = \partial_u x \partial_v y - \partial_v x \partial_u y$ is the determinant of the Jacobian matrix. Here D is a region in the xy plane, and D^* is the same region expressed in terms of u and v.

1.3.1 Aside on Jacobians and orientation

It is important to note that if $(x, y) \mapsto (u, v)$ is an invertible change of variables, meaning that the functions $x(u, v)$, $y(u, v)$, $u(x, y)$, and $v(x, y)$ all exist and are differentiable throughout the regions of integration, then we cannot have $|\partial(x, y)/\partial(u, v)| = 0$ anywhere, and so we must have $\partial(x, y)/\partial(u, v) > 0$ or $\partial(x, y)/\partial(u, v) < 0$ everywhere.

In the former case we say the change of variables is *orientation preserving* and we may write (3) as

$$\iint_D f\,dxdy = \iint_{D^*} f \frac{\partial(x, y)}{\partial(u, v)} dudv, \tag{4}$$

and in the latter case we say the change of variables is *orientation reversing* and we may write (3) as

$$\iint_D f\,dxdy = -\iint_{D^*} f \frac{\partial(x, y)}{\partial(u, v)} dudv. \tag{5}$$

Note that switching u and v makes an orientation preserving change of variables into an orientation reversing one, and vice versa.

2 Differential forms

Two dimensional *differential forms* come in three sorts: zero forms, one forms, and two forms. The number zero, one, or two, is the *degree* of the form.

2.1 Zero forms

A *zero form* is an expression f, where f is a function $f(x, y)$; this is just a new name for a familiar object.

2.2 One forms

A *one form* is an expression

$$F_1 dx + F_2 dy,$$

where $F_1 = F_1(x, y)$ and $F_2 = F_2(x, y)$. Considering such a one form is equivalent to considering the corresponding vector field (F_1, F_2); we can think of this as alternative notation akin to writing $(2, 3) = 2\hat{\imath} + 3\hat{\jmath}$.

We define an operation d, the *total differential* or *exterior derivative*, from zero forms to one forms, by

$$df = \partial_x f dx + \partial_y f dy. \tag{6}$$

Then (1) can be rewritten as

$$\int_C df = f(q) - f(p),$$

and we can change variables as follows:

$$\begin{aligned} F_1 dx + F_2 dy &= F_1(\partial_u x du + \partial_v x dv) + F_2(\partial_u y du + \partial_v y dv) \\ &= (F_1 \partial_u x + F_2 \partial_u y) du + (F_1 \partial_v x + F_2 \partial_v y) dv, \end{aligned} \tag{7}$$

where we used some of the same notation as in §1.3.

2.3 Two forms

A *two form* is an expression

$$f dx \wedge dy,$$

where $f = f(x, y)$. The symbol $\wedge$ is pronounced 'wedge'.

We define the double integral of a two form by just deleting the wedge:

$$\iint_D f dx \wedge dy = \iint_D f dxdy \tag{8}$$

We extend the operation d from (6) to take one forms to two forms by putting

$$d(F_1 dx + F_2 dy) = (\partial_x F_2 - \partial_y F_1) dx \wedge dy, \tag{9}$$

so that in particular we have

$$d(df) = d(\partial_x f dx + \partial_y f dy) = (\partial_x \partial_y f - \partial_y \partial_x f) dx \wedge dy = 0. \tag{10}$$

Then if we let α be a one form, we see that (2) becomes

$$\iint_D d\alpha = \int_{\partial D} \alpha.$$

2.4 Wedge

The operation $\wedge$ extends to a more general product on differential forms as follows. If f and g are zero forms, then $f \wedge g$ is just the usual product fg. Similarly if f is a zero form and $F_1 dx + F_2 dy$ is a one form, then

$$f \wedge (F_1 dx + F_2 dy) = f F_1 dx + f F_2 dy.$$

Wedging together two one forms is more interesting: we put

$$(F_1 dx + F_2 dy) \wedge (G_1 dx + G_2 dy) = (F_1 G_2 - F_2 G_1) dx \wedge dy. \tag{11}$$

Note the following things:

1. There is a close correspondence between $\wedge$ and cross product:

$$(F_1, F_2, 0) \times (G_1, G_2, 0) = (0, 0, F_1G_2 - F_2G_1),$$

   and an analogous one between d and curl:

$$\mathrm{curl}(F_1, F_2, 0) = (0, 0, \partial_x F_2 - \partial_y F_1).$$

   Equation (10) above corresponds to the fact that the curl of a gradient is zero.

2. The wedge product is *antisymmetric* on one forms:

$$(F_1dx + F_2dy) \wedge (G_1dx + G_2dy) = -(G_1dx + G_2dy) \wedge (F_1dx + F_2dy). \tag{12}$$

3. We could equivalently define d on one forms by

$$d(F_1dx + F_2dy) = dF_1 \wedge dx + dF_2 \wedge dy. \tag{13}$$

To check (13), apply (9) to the left hand side and (6) and (11) to the right hand side. Actually, (13) is a special case of the more general *product rule*:

$$d(f\alpha) = df \wedge \alpha + fd\alpha,$$

for any zero form f and one form α.

Combining (7) and (11) gives a change of variables formula for two forms:

$$\begin{aligned} f dx \wedge dy &= f(\partial_u x du + \partial_v x dv) \wedge (\partial_u y du + \partial_v y dv) \\ &= f(\partial_u x \partial_v y - \partial_v x \partial_u y) du \wedge dv. \end{aligned}$$

Integrating both sides and using (8) gives us a variant of (3)

$$\iint_D f dx dy = \iint_D f(\partial_u x \partial_v y - \partial_v x \partial_u y) du \wedge dv. \tag{14}$$

To get (3) from this one keeps track of the orientation in the conversion from an integral over D to an integral over D^*.

2.5 Hodge star

We have seen how $\wedge$ on one forms corresponds to taking a cross product, and d to taking a curl. To get operations corresponding to dot product and divergence, we introduce the *Hodge star* operator, denoted $*$.

The Hodge star operator is defined by

$$*(F_1dx + F_2dy) = F_1dy - F_2dx. \tag{15}$$

It corresponds to a right angle rotation of the vector (F_1, F_2). Combining with $\wedge$ gives

$$(F_1 dx + F_2 dy) \wedge *(G_1 dx + G_2 dy) = (F_1 G_1 + F_2 G_2) dx \wedge dy,$$

which corresponds to the dot product $(F_1, F_2) \cdot (G_1, G_2)$. Similarly we have

$$d * (F_1 dx + F_2 dy) = (\partial_x F_1 + \partial_y F_2) dx \wedge dy,$$

which corresponds to the divergence $\operatorname{div}(F_1, F_2)$.

The Hodge star converts the usual statement of Green's Theorem (2) to a version related to the divergence:

$$\iint_D (\partial_x F_1 + \partial_y F_2) dA = \int_{\partial D} F_1 dy - F_2 dx.$$

The integral on the right can be interpreted as the flux of the vector field (F_1, F_2) outward through the boundary of D, as can be seen by writing it out in terms of a parametrization and observing that the integrand is the dot product of the vector field with an outward normal vector.

3 Further reading

The references below are to the sixth edition of the book *Vector Calculus* by Marsden and Tromba, and to Arapura's notes on differential forms available at `https://www.math.purdue.edu/~dvb/preprints/diffforms.pdf`.

For more on the line integral version of the Fundamental Theorem of Calculus, see sections 7.2 of the book and 1.4 of the notes. For more on Green's Theorem, see section 8.1 of the book and 1.6 of the notes. For more on changes of variables, see section 6.2 of the book. For more on two forms, see section 2 of the notes.

A more complete account of this material can be found in the book *Calculus on Manifolds* by Spivak.