MA 366 final review problems

Hopefully final version as of April 24th.

- The final will be on Friday, May 3rd, from 7 to 9 pm in WTHR 104.
- It will cover the material from the whole semester.
- No notes or books or electronic devices are allowed, but the reference page at the back of this document will be included on the exam.
- Most of the problems on the final will be closely based on the following problems, and on the review and exam problems from the midterms.
- For each problem, you must justify your answers.
- Please let me know if you have a question or find a mistake, and note that these are not arranged in order of difficulty!

1. Sketch a phase portrait for the systems below near each equilibrium.

   (a)
   $$x' = xy - 1,$$
   $$y' = xy - 1 + y - x.$$

   (b)
   $$x' = 1 - xy,$$
   $$y' = xy - 1 + y - x.$$

   (c)
   $$x' = \sin(\pi y) - x,$$
   $$y' = xy - y^2 + y.$$

   Hint: Each system has two equilibiria.

2. Let a be a real number. For each value of a, decide whether the system below has a stable, asymptotically stable, or unstable equilibrium at the origin:
   $$x' = y + ax(1 + y^2),$$
   $$y' = -2x^3 + ay(1 + x^2).$$
   Hint: Let $V(x, y) = x^4 + y^2$ and consider $\frac{d}{dt}V(x(t), y(t))$.

3. The van der Pol oscillator equation is
   $$x'' + \varepsilon(x^2 - 1)x' + x = 0,$$
   where ε is a real parameter.

(a) Let $y = x'$ and rewrite the equation as a first order system in x and y.

(b) Find the equilibrium points of this system and sketch a phase portrait near each one. How does the answer depend on ε?

4. Consider the system

$$x' = -x^3 - xy^{2k},$$
$$y' = -y^3 - x^{2k}y,$$

where k is a given positive integer.

(a) Find and classify according to stability the equilibrium solutions.

Hint: Let $V(x, y) = x^2 + y^2$ and consider $\frac{d}{dt}V(x(t), y(t))$.

(b) Sketch a phase portrait when $k = 1$.

Hint: What are x' and y' when $y = ax$ for some real number a?

5. The motion of a forced spring mass system is given by

$$y''(t) + y(t) = au(t-4) - au(t-b), \qquad y(0) = y'(0) = 0.$$

for some $a > 0$ and $b > 4$, where $u(t) = 1$ when $t \geq 0$ and $u(t) = 0$ when $t < 0$. Use the Laplace transform to find all values of a and b such that $y(t) = 0$ for $t > b$.

6. A population of bacteria, with harvesting beginning at time $T > 0$, is given by

$$y'(t) = 2y(t) - 1000u(t-T), \qquad y(0) = 10,$$

where $u(t) = 1$ when $t \geq 0$ and $u(t) = 0$ when $t < 0$.

(a) Use the Laplace transform to find $y(t)$. (You may enjoy also using integrating factors to do this.)

(b) Find all values of T such that the population remains always positive.

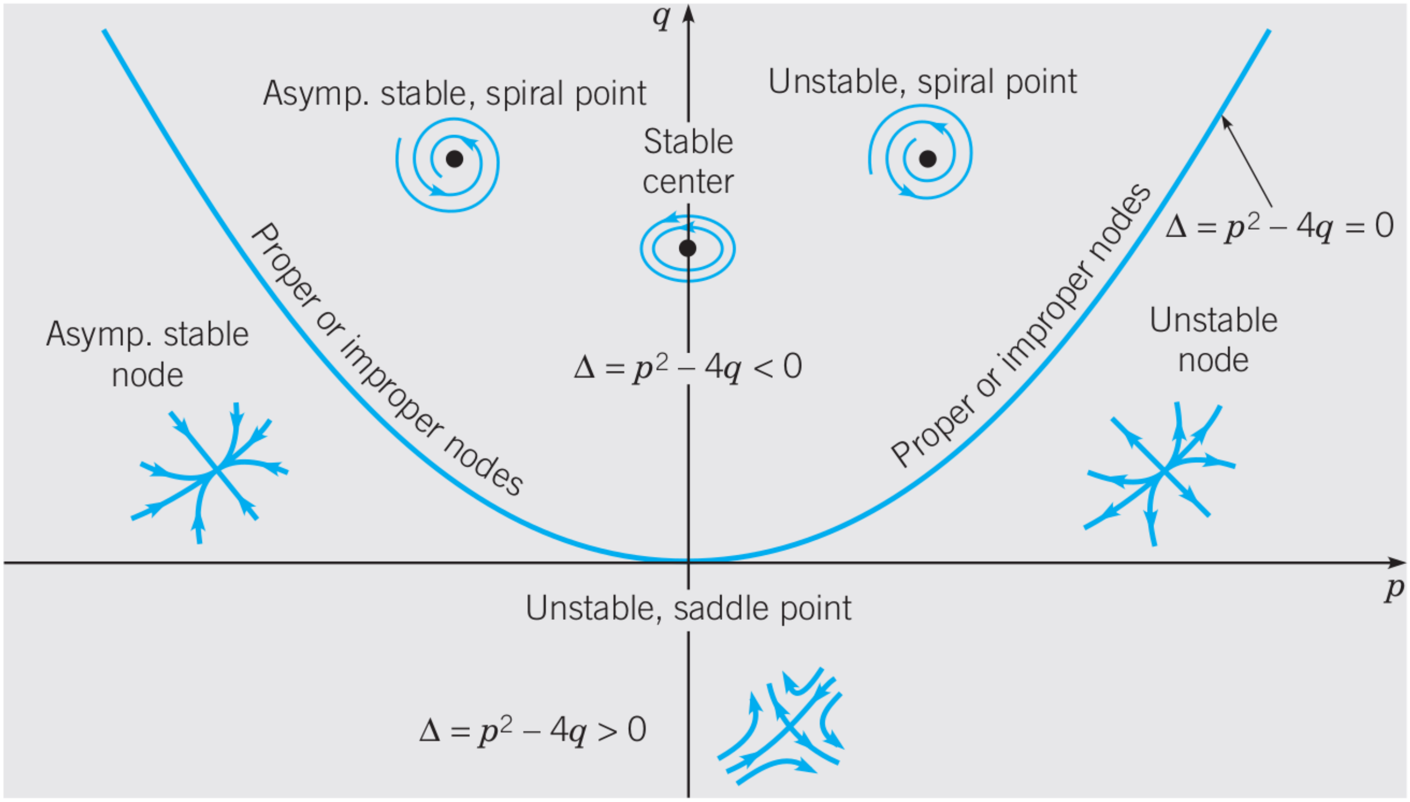
(c) If T is the smallest of the values you found in part (b), what is the maximum population and at which times is it achieved?

7. Solve

$$y''(t) + y(t) = 3\delta(t-1) + a\delta(t-b), \qquad y(0) = y'(0) = 0,$$

where a is any real number and $b > 1$. Find all a and b such that $y(t) = 0$ for all $t > 10$.

Reference page



Here $p = \text{trace} = a + d = \lambda_1 + \lambda_2$ and $q = \text{determinant} = ad - bc = \lambda_1 \lambda_2$.

If A is defective, then $x' = Ax$ is solved by

$$x(t) = e^{\lambda t}(C_1 v + C_2(tv + w)),$$

where $(A - \lambda)v = 0$ and $(A - \lambda)w = v$.

The Laplace transform is given by

$$\mathcal{L}[f] = F(s) = \int_0^\infty e^{-ts} f(t) dt.$$

We have $\mathcal{L}[e^{at}] = 1/(s - a)$, $\mathcal{L}[\sin bt] = b/(s^2 + b^2)$, $\mathcal{L}[\cos bt] = s/(s^2 + b^2)$, $\mathcal{L}[t^n] = n!/s^{n+1}$, and

$\mathcal{L}[f(t - a)u(t - a)] = e^{-as}\mathcal{L}[f]$, $\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)$, $\mathcal{L}[f''] = s^2\mathcal{L}[f] - sf(0) - f'(0)$.

$\mathcal{L}[\delta(t - a)] = e^{-as}$.