MA 366 midterm review problems

Hopefully final version as of January 30th.

- The midterm will be on Wednesday, February 6th, from 8 to 9 pm in ME 1130.
- It will cover all the material from the classes and homework up to that date.
- No notes or books or electronic devices are allowed.
- Most of the problems on the exam will be closely based on ones from the final version of the list below (but of course the actual exam will be much shorter).
- For each problem, you must justify your answers.
- Please let me know if you have a question or find a mistake, and note that these are not arranged in order of difficulty!

1. Consider the differential equation
$$t^2y'' + aty' + y = 0, \qquad t > 0,$$
where a is a given real number. For which values of r is $y(t) = t^r$ a solution? (The answer depends on a.)

2. Find the general solution to
$$y' = xy + \frac{y}{x+1}.$$

3. Solve the initial value problem
$$y' = xy^2, \qquad y(0) = a,$$
where a is a given real number. What is the interval of existence? (The answer depends on a.)

4. Give an interval in which the solution to the initial value problem
$$(t^4 - 16)y' + \cos(\sin(e^t))y = \ln(1 + e^t), \qquad y(\pi) = e,$$
is guaranteed to exist. How big of an interval can you get?

5. A snowball is melting such that the rate of decrease of the volume is proportional to the 2/3 power of the volume. If the snowball is half as big after 5 minutes, how long does it take to melt away completely?

6. (a) Sketch a phase diagram for $x' = x^3 - 2x^2 + x$. Label all equilibrium solutions as stable, unstable, or semi-stable.

(b) How does the phase diagram change if we consider instead $x' = x^3 - 2x^2 + x + 0.0001$? Do not try to compute the equilibrium solutions precisely, but instead describe how they are related to the ones in part (a).

(c) How does the phase diagram change if we consider instead $x' = x^3 - 2x^2 + x - 0.0001$? Again do not try to compute the equilibrium solutions precisely, but instead describe how they are related to the ones in part (a).

(d) For each of the above parts, sketch typical solutions of the equation.

(e) For each of the above parts, if $x(0) = 1$, is $\lim_{t\to\infty} x(t)$ finite or infinite, and what can you say about its value?

7. Find the equilibrium solutions to the differential equation

$$y' = e^y(y^3 + 5y^2 + 6y)$$

and classify them according to stability.

8. Find the equilibrium solutions to the differential equation

$$y' = y^2 - y^4$$

and classify them according to stability.

9. Let $f(x, y) = 2x^2 + 3\sin 4y$ and consider the initial value problem

$$y'(x) = f(x, y(x)), \qquad y(0) = 0.$$

(a) Let Picard iterates be defined by

$$y_0 \equiv 0, \qquad y_m(x) = \int_0^x f(s, y_{m-1}(s))ds.$$

Find $y_1(x)$ and $y_2(x)$. You may leave an unevaluated definite integral in your answer for $y_2(x)$.

(b) We saw that

$$|y(x) - y_m(x)| \leq 2^{1-m} \max_{|s|\leq\frac{1}{2M}} |y_0(s) - y_1(s)|,$$

as long as $|x| \leq \frac{1}{2M}$ and $|\partial_y f| \leq M$.
Find the best value of M you can use in this situation. Then substitute $m = 0$, $m = 1$, and $m = 2$ into the formula and simplify the right hand side as much as possible in each case.

10. Use the substitution $u = y^{-2}$ to find the general solution to the differential equation

$$\frac{dy}{dx} + xy + x^3y^3 = 0.$$

Solve for y in your answer.

11. Use the substitution $y(t) = u(t)t^{-2}$ to find the general solution to the differential equation

$$t^2y'' + 5ty' + 4y = 0, \qquad t > 0.$$

12. Solve the following initial value problems. What is the interval of existence in each case?

(a)

$$y' = 3y^2, \qquad y(6) = 2,$$

(b)

$$xy' = 3y + 4x^{-1}, \qquad y(1) = 2,$$

(c)

$$xy' = 3y + 4x, \qquad y(1) = 2,$$

(d)

$$x^2y' + xy = \sin x, \qquad y(1) = 2,$$

(for (d) you may leave an unevaluated definite integral in your answer)

(e)

$$y' = (x + y + 2)^2, \qquad y(-2) = 0.$$

(for (e) you might try the substitution $u = x + y + 2$),

(f)

$$xy'' + 2y' = 6x, \qquad y(10) = y'(10) = 0,$$

(for (f) you might try the substitution $u = y'$).

13. Let a and b be real numbers, and let m and n be natural numbers. Suppose the differential equation

$$y' = ay^m(y-b)^n$$

produces the slope field below.

(a) Find b.

(b) Is a positive or negative?

(c) Is m even or odd?

(d) Is n even or odd?