Reduction of order

Consider the differential equation

$$y'(t) = 3y(t). \tag{1}$$

A solution to this equation is given by e^{3t}. The method of reduction of order consists of making the substitution

$$y(t) = u(t)e^{3t}, \tag{2}$$

where $u(t)$ is an unknown function to be determined. Substituting equation (2) into equation (1) and using the product rule gives

$$u'(t)e^{3t} + 3u(t)e^{3t} = 3u(t)e^{3t}.$$

This simplifies to

$$u'(t) = 0,$$

which means $u(t) = C$ for some constant C. Plugging back into equation (2) gives

$$y(t) = Ce^{3t}.$$

We say that e^{3t} is a *particular* solution to (1). We say that Ce^{3t}, where C is any constant, is the *general* solution to (1).

This method generalizes to more complicated problems. Let us now consider the differential equation

$$y'(t) = 3y(t) + 6. \tag{3}$$

We use the same substitution given in equation (2) to get

$$u'(t)e^{3t} + 3u(t)e^{3t} = 3u(t)e^{3t} + 6,$$

which simplifies to

$$u'(t) = 6e^{-3t}.$$

Integrating gives

$$u(t) = -2e^{-3t} + C,$$

for some constant C. Plugging back into equation (2) gives

$$y(t) = 2 + Ce^{3t},$$

as the general solution to (3).