1. Discrete Fourier Transform

1.1. Motivation. The Fourier series (and later, Fourier transform) is often used to analyze continuous periodic signals. However, in practice, the signal is often a discrete set of data. The discrete Fourier transform is the discrete analogue of Fourier series/transform. It can be obtained by discretizing the Fourier series as follows: Consider a continuous function $f$ with period $2\pi$, which can be viewed as a function on $[0, 2\pi]$. Fix an integer $N$, we can make a partition of the interval $[0, 2\pi]$ into $N$ subintervals of equal size $[x_j, x_{j+1}]$, $j = 0, 1, \ldots, N-1$ where $x_j = jh$ and $h = 2\pi/N$. The discretization of the function $f$ gives the finite sequence of numbers $y = \{y_j = f(x_j)\}_{j=0}^{N-1}$ (in signal process, this process is called sampling). The $k$-th Fourier coefficient of $f$ is given by

$$\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-ikx}dx, \quad k \in \mathbb{Z}. \quad (1.1)$$

The Riemann integration tells us that we can use the following sum to approximate the integration

$$\hat{f}_k \approx \frac{1}{2\pi} \sum_{j=0}^{N-1} hf(x_j)e^{-ikx_j} = \frac{1}{N} \sum_{j=0}^{N-1} y_j e^{-i2\pi jk/N} := \frac{1}{N} \hat{y}_k. \quad (1.2)$$

We can of course extend the sequence $y = \{y_j\}$ to an infinite sequence of period $N$, then the above expression gives another infinite sequence $\hat{y} = \{y_k\}$ of period $N$ since $e^{2\pi i} = 1$. In particular, if we regard both $y$ and $\hat{y}$ as vectors in $\mathbb{C}^N$, this transformation $y \mapsto \hat{y}$ can be realized by a matrix

$$F_N = (e^{-i2\pi jk/N})_{j,k=0,1,\ldots,N-1} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{pmatrix}, \quad \omega = e^{-2\pi i/N}. \quad (1.3)$$
For the discrete signal data \( y = (y_j) \), we can view the components of \( \hat{y} = (\hat{y}_k) \) as the strength of the different frequency.

**Remark 1.1.** The matrix \( F_N \) is an example of a Vandermonde matrix

\[
A = \begin{pmatrix}
1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\
1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\
1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_n & a_n^2 & \cdots & a_n^{n-1}
\end{pmatrix}
\]  

(1.4)

with \( a_j = \omega^{j-1} \). The determinant of (1.4) is given by

\[
\det(A) = \prod_{1 \leq j < k \leq n} (a_k - a_j).
\]

Therefore we have a (not very useful) formula for \( \det(F_N) \)

\[
\det(F_N) = \prod_{0 \leq j < k \leq N-1} (\omega^k - \omega^j) = \prod_{0 \leq j < k \leq N-1} \omega^j(\omega^{k-j} - 1) = \prod_{j=0}^{N-2} \omega^{j(N-1-j)} \prod_{\ell=1}^{N-1} (\omega^\ell - 1)^{N-\ell}.
\]

Later on, we see that \( \det(F_N) = N^{N/2} \) up to a unitary factor.

1.2. **Definition.** Now we define the discrete Fourier transform \( \mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N \). We denote the vectors in \( \mathbb{C}^N \) by \( u = (u_j)_{j=0}^{N-1} \) and define its discrete Fourier transform \( \hat{u} = \mathcal{F}_N u \) by

\[
\hat{u}_k = (\mathcal{F}_N u)_k = \sum_{j=0}^{N-1} e^{-i2\pi jk/N} u_j.
\]

(1.5)

In particular, \( \mathcal{F}_N \) can be viewed as the \( N \times N \) matrix (1.3):

\[
\mathcal{F}_N = (e^{-i2\pi jk/N})_{j,k=0,1,\ldots,N-1}.
\]

Let \( \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} = \{0,1,\ldots,N-1\} \), then we can view a vector \( u \in \mathbb{C}^N \) as a function on \( \mathbb{Z}_N \) with value \( u_j \) at \( j \in \mathbb{Z}_N \). The length of the vector \( u \) is the analogue of the \( L^2 \)-norm of a function

\[
\|u\| = \left( \sum_{j=0}^{N-1} |u_j|^2 \right)^{1/2}.
\]

(1.6)

There is also a natural inner product on \( \mathbb{C}_N = \ell^2(\mathbb{Z}_N) \):

\[
(u, v) = \sum_{j=0}^{N-1} u_j \bar{v}_j.
\]

(1.7)
We denote \( \zeta = \omega^{-1} = e^{2\pi i/N} \) and define the vectors \( \chi^{(k)} \in \mathbb{C}^N, k = 0, 1, \ldots, N - 1 \) to be
\[
\chi^{(k)}_j = \zeta^j = e^{i2\pi jk/N}, \quad j = 0, 1, \ldots, N - 1, \tag{1.8}
\]
then we see
\[
\hat{u}_k = (\mathcal{F}_Nu)_k = (u, \chi^{(k)}).
\]

As explained in the motivation, we can also view \( \mathcal{F}_N \) as a linear transformation on the vector space \( S_N \) of sequences of complex numbers \( \{y_j\}_{j \in \mathbb{Z}} \) with periodic \( N \).

Remark 1.2. The set \( \mathbb{Z}_N \) has a structure of an “abelian group” under addition modulo \( N \) and the functions \( \chi^{(k)} \) are “characters” of \( \mathbb{Z}_N \), i.e. group homomorphism \( G \to S^1 \) where \( S^1 \) is the circle group \( \{ z \in \mathbb{C} : |z| = 1 \} \) under multiplication. We can generalize this definition to any finite abelian group \( G \) and get the Fourier transform on \( G \) (see [SS, Chapter 7]) or even infinite abelian groups (LCA group, see [D, Part II]) like \( \mathbb{R} \) or \( S^1 = \mathbb{R}/2\pi\mathbb{Z} \). This provides a uniform point of view for the Fourier series
\[
f(\theta) \in L^2(S^1) \mapsto \left\{ \hat{f}_k = \frac{1}{2\pi} \int_{S^1} f(\theta)e^{-ik\theta} d\theta \right\} \in \ell^2(\mathbb{Z}),
\]
(and its inverse, Z-transform), the Fourier transform
\[
f(x) \in L^2(\mathbb{R}) \mapsto \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx,
\]
and discrete Fourier transform
\[
(u_j) \in \ell^2(\mathbb{Z}_N) \mapsto \left\{ \hat{u}_k = \sum_{j=0}^{N-1} u_j e^{-2\pi ijk/N} \right\} \in \ell^2(\mathbb{Z}_N).
\]

1.3. Basic properties. Now we present the basic properties of the discrete Fourier transform \( \mathcal{F}_N \):

Theorem 1.3. The vectors \( \chi^{(k)}, k = 1, \ldots, N \) are orthogonal to each other and more precisely
\[
(\chi^{(k)}, \chi^{(\ell)}) = N\delta_{k\ell},
\]
where \( \delta_{k\ell} \) is the Kronecker symbol: \( \delta_{k\ell} = 1 \) if \( k = \ell \), 0 otherwise. In particular, \( \left\{ \frac{1}{\sqrt{N}} \chi^{(k)} \right\} \)
is an orthonormal basis of \( \mathbb{C}^N \) and the matrix \( U_N := \frac{1}{\sqrt{N}} \mathcal{F}_N \) is a unitary matrix:
\[
F_N F_N^* = F_N^* F_N = NI_N.
\]
Here \( F_N^* \) is the adjoint matrix of \( F_N \) and \( I_N \) is the identity \( N \times N \) matrix. As a corollary, the inverse discrete Fourier transform \( \mathcal{F}_N^{-1} \) is given by the matrix \( F_N^{-1} = N^{-1} F_N^* \), i.e.
\[
u_j = \frac{1}{N} \sum_{k=0}^{N-1} \hat{u}_k e^{i2\pi jk}.
\]
Moreover, for any $u, v \in \mathbb{C}^N$, we have the Plancherel theorem
\[ (\mathcal{F}_N u, \mathcal{F}_N v) = N(u, v) \] (1.9)
and in particular, the Parseval identity
\[ \|\mathcal{F}_N u\|^2 = N\|u\|^2. \]

**Remark 1.4.** Sometimes people prefer to use $\mathcal{U}_N := \frac{1}{\sqrt{N}}\mathcal{F}_N$ instead of $\mathcal{F}_N$ as the discrete Fourier transform with the advantage of being unitary.

**Proof.** We can compute directly that
\[ (\chi^{(k)}, \chi^{(\ell)}) = \sum_{j=0}^{N-1} \zeta^{jk} \bar{\zeta}^j \ell = \sum_{j=0}^{N-1} \zeta^{j(k-\ell)}. \]

When $k = \ell$, clearly we have $(\chi^{(k)}, \chi^{(\ell)}) = N$ since all the terms are 1. When $k \neq \ell$, this is a geometric series and we have
\[ (\chi^{(k)}, \chi^{(\ell)}) = \frac{\zeta^{(k-\ell)N} - 1}{\zeta^{k-\ell} - 1} = 0 \]
since $\zeta^N = 1$. All of the remaining statements follow by simple linear algebra. \qed

1.4. **Examples.**

**Example 1.5.** For $N = 2$, we have
\[ F_N = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad F^{-1}_N = \frac{1}{2} F_N = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \]

For $N = 4$, we have
\[ F_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}, \quad F^{-1}_N = \frac{1}{4} F^*_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -i & 1 & -1 \\ 1 & -1 & 1 & -i \end{pmatrix}. \]

**Example 1.6.** Let $\delta_0 \in \ell^2(\mathbb{Z}_N)$ be given by $\delta_0 = (\delta_{0j}) = (1, 0, \ldots, 0)$, then
\[ (\hat{\delta}_0)_k = 1, \quad k \in \mathbb{Z}_N \]
and thus
\[ \delta_0 = \frac{1}{N} \sum_{k=0}^{N-1} \chi^{(k)}. \]
More generally, if $u$ is the square wave centered at 0 with length $W$ (an odd integer smaller than $N$)

$$u_j = \begin{cases} 
1/W & \text{for } j = 0, 1, \ldots, \left[ \frac{W-1}{2} \right] \text{ or } j = N - \left[ \frac{W-1}{2} \right], \ldots, N - 1; \\
0 & \text{otherwise},
\end{cases}$$

then

$$\hat{u}_k = \begin{cases} 
1 & \text{if } k = 0; \\
\frac{\sin(\pi W_k/N)}{\sin(\pi k/N)} & \text{otherwise}
\end{cases}$$

1.5. More properties. Now we discuss the interaction between discrete Fourier transform $F_N$ with other basic operations on $\mathbb{C}^N$.

First we consider the circular shift operator $\tau_m$ on $\mathbb{C}^N = \ell^2(\mathbb{Z}_N)$, which is an analogue of the translation operator, and is defined by

$$(\tau_m u)_j = u_{j-m}, \quad j \in \mathbb{Z}_N$$

(1.10)

where we understand $j - m$ as modulo $N$. For example, if $u = (u_0, \ldots, u_{N-1})$, then

$$\tau_1 u = (u_{N-1}, u_0, \ldots, u_{N-2}),$$

and the matrix associated to $\tau_1$ is

$$\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}$$

If we think $u \in \mathbb{C}^N$ as a period-$N$ sequence $(u_j)_{j \in \mathbb{Z}} \in S_N$, then the circular operator $\tau_m$ is exactly the translation operator $(\tau_m u)_j = u_{j-m}, \ j \in \mathbb{Z}$.

Proposition 1.7. For any $m \in \mathbb{Z}$, and any $u \in \mathbb{C}^N$,

$$[F_N(\tau_m u)]_k = e^{-2\pi i km/N} \hat{u}_k$$

(1.11)

and

$$F_N(\{e^{2\pi i jm/N} u_j\}) = \tau_m \hat{u}.$$  

(1.12)

Remark 1.8. This means that the discrete Fourier transform $F_N$ diagonalizes the matrix of $\tau_m$. The matrix of $\tau_m$ is an example of a circulant matrix, i.e. the entries on each of
the generalized diagonals are constant. In an $N \times N$ matrix $(a_{jk})$, a generalized diagonal is formed by the elements $a_{jk}$ with $j - k$ a constant modulo $N$. For example,

$$
\begin{pmatrix}
1 & 2 & 3 & 4 \\
4 & 1 & 2 & 3 \\
3 & 4 & 1 & 2 \\
2 & 3 & 4 & 1
\end{pmatrix}
$$

In fact, $\mathcal{F}_N$ diagonalizes all circulant matrices.

We can also define the convolution $u \ast v$ of $u, v \in \mathcal{S}_N$ by

$$
[u \ast v]_k = \sum_{j=0}^{N-1} u_j v_{k-j}.
$$

(1.13)

This also gives a convolution on $\mathbb{C}^N$ by interpreting the indices $k - j$ as modulo $N$.

**Proposition 1.9.** For any $u, v \in \mathbb{C}^N$,

$$
\mathcal{F}_N[u \ast v] = (\mathcal{F}_N u) \cdot (\mathcal{F}_N v)
$$

(1.14)

where the right-hand side is the multiplication in each entry. Similarly,

$$
\mathcal{F}_N[u \cdot v] = \frac{1}{N} (\mathcal{F}_N u) \ast (\mathcal{F}_N v).
$$

(1.15)

Finally, we define the reverse $\mathcal{R}u$ of $u \in \mathbb{C}^N$ by

$$(\mathcal{R}u)_k = u_{N-k}$$

(1.16)

where again, we understand $N - k$ modulo $N$ and thus $(\mathcal{R}u)_0 = u_0$. We also define $\bar{u} \in \mathbb{C}^N$ to be the complex conjugate of $u$ by taking complex conjugate of each entry.

**Proposition 1.10.** For any $u \in \mathbb{C}^N$,

$$
\mathcal{F}_N(\mathcal{R}u) = \mathcal{R}(\mathcal{F}_N u) = N\mathcal{F}_N^{-1}(u);
$$

(1.17)

$$
\mathcal{F}_N(\bar{u}) = N\mathcal{F}_N^{-1}(u), \quad \mathcal{F}_N(u) = N\mathcal{F}_N^{-1}(\bar{u}).
$$

(1.18)

1.6. **Eigenvalues and eigenvectors.** The unitary discrete Fourier transform $\mathcal{U}_N = \frac{1}{\sqrt{N}} \mathcal{F}_N$ satisfies

$$
\mathcal{U}_N^4 = I.
$$

In other words, if we apply $\mathcal{U}_N$ to a vector four times, we get the original vector. To see this, since $\mathcal{R}^2 u = u$, we can use (1.29) for $\mathcal{R}u$ to get $\mathcal{F}_N u = N\mathcal{F}_N^{-1}(\mathcal{R}u)$, and thus

$$
\mathcal{F}_N^2 u = \mathcal{F}_N(N\mathcal{F}_N^{-1}(\mathcal{R}u)) = N\mathcal{R}u.
$$
Therefore the eigenvalues of $U_N$ are (included in) $\pm 1, \pm i$, each possibly with multiplicity. The characteristic polynomial of $U_N$ is actually given by
\[
\det(\lambda I - U_N) = (\lambda - 1)^{\frac{N}{4}} + 1 (\lambda + 1)^{\frac{N+2}{4}} (\lambda + i)^{\frac{N+1}{4}} (\lambda - i)^{\frac{N-1}{4}}.
\]
However, due to multiplicity, there are no simple formula for the general eigenvectors. Many different choices of a basis of eigenvectors are proposed.

1.7. **Fast Fourier transform.** Since discrete Fourier transform is such a valuable tool in application, it is important to develop efficient algorithms to compute discrete Fourier transform and inverse, called fast Fourier transform. The idea of such algorithms were popularized in 1965 by Cooley and Tukey, but can be traced back to Gauss in 1805. The naive way to compute the discrete Fourier transform $F_N$ of a vector $u \in \mathbb{C}^N$ requires $O(N^2)$ arithmetic operations, since we are multiplying a $N \times N$ matrix $F_N$ with a $N$-vector $u$. However, by exploiting the special structure of the matrix $F_N$, we can reduce to $O(N \log N)$ operations. As an example, we discuss the simplest case where $N = 2^n$ is a power of 2.

**Theorem 1.11.** Given $\omega_N = e^{-2\pi i/N}$ with $N = 2^n$, it is possible to calculate $F_N u$ with at most $3N \log_2(N) = O(N \log N)$ calculations.

The key idea is to use $F_M$ to calculate $F_{2M}$.

**Lemma 1.12.** Let $\#(M)$ denote the minimum number of operations needed to calculate $F_M u$ for any $u \in \mathbb{C}^M$. If we are given $\omega_{2M} = e^{-2\pi i/(2M)} = e^{-\pi i/M}$, then
\[
\#(2M) \leq 2\#(M) + 6M.
\]

**Proof.** Let $u = (u_0, u_1, \ldots, u_{2M-1}) \in \mathbb{C}^{2M}$, we split $u$ into two vectors $u_{\text{even}}$ and $u_{\text{odd}}$ in $\mathbb{C}^M$ by taking only the even subindices or the odd subindices:
\[
u_{\text{even}} = (u_0, u_2, \ldots, u_{2M-2}), \quad u_{\text{odd}} = (u_1, u_3, \ldots, u_{2M-1}).
\]
By definition, and $\omega_{2M}^2 = \omega_M$,
\[
(F_{2M} u)_k = \sum_{j=0}^{2M-1} \omega_{2M}^j u_j = \sum_{j=0}^{M-1} \omega_M^j u_{2j} + \omega_{2M}^k \sum_{j=0}^{M-1} \omega_M^j u_{2j+1} = (F_M u_{\text{odd}})_k + \omega_{2M}^k (F_M u_{\text{even}})_k
\]
Therefore to compute $F_{2M} u$, we simply need to compute $F_M u_{\text{odd}}$ and $F_M u_{\text{even}}$ and all of $\omega_{2M}^k$ with $k = 1, 2, \ldots, 2M - 1$. Notice that this also gives all $\omega_M^k$ with $k = 1, 2, \ldots, M - 1$. We have
\[
\#(2M) \leq 2\#(M) + 2M + 2 \cdot 2M = 2\#(M) + 6M.
\]
\[\square\]
Now we can finish the proof of the theorem by induction. For \( n = 1, N = 2 \), we have
\[
(F_2 u)_0 = u_0 + u_1, \quad (F_2 u)_1 = u_0 - u_1
\]
which requires at most 3 operations which is less than 6. Now suppose for \( N = 2^{n-1} \), we can compute \( F_{N/2} u \) for any \( u \in \mathbb{C}^{N/2} \) with at most \( 3(N/2) \log_2(N/2) \) operations. We consider the case \( N = 2^n \) and by the lemma, we need at most
\[
2 \cdot 3 \frac{N}{2} \log_2(N/2) + 6 \cdot \frac{N}{2} = 3N \log_2 N
\]
operations. By induction we finish the proof.

**Remark 1.13.** The same procedure can be developed for inverse discrete Fourier transform \( F_N^{-1} \) which also only requires \( O(N \log N) \) operations. More generally, we can develop similar procedure for \( N \) of the form \( N = p^n \) or even \( N = N_1 N_2 \cdots N_n \). Other fast Fourier transforms have been developed to deal with general \( N \), e.g. Rader’s algorithm.

**Remark 1.14.** It is unknown whether we can beat the \( O(N \log N) \) time required in the Cooley–Tukey method above.

### 1.8. Comparing to the continuous Fourier transform.

The Fourier transform for a function \( f \) on \( \mathbb{R} \) is defined by
\[
(F f)(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x) dx.
\]

**Remark 1.15.** This choice of the phase with factor \( 2\pi \) in the definition makes \( F \) a unitary operator as we will see later in the Parseval identity. Sometimes people use the definition without \( 2\pi \) in the phase, then we also need to multiply by \( 1/\sqrt{2\pi} \) to make it unitary.

To make sense of this integration, we have to impose certain assumptions on the function \( f \). For example, this integral converges for any Lebesgue integrable function \( f \in L^1(\mathbb{R}) \). However, to study the properties of Fourier transform, it is better to concentrate on a special class of functions first, called the Schwartz class, often denoted by \( \mathcal{S}(\mathbb{R}) \). A function \( f \) is said to belong to \( \mathcal{S}(\mathbb{R}) \), if all of its derivatives are rapidly decreasing as \( |x| \to \infty \), in the sense that for any integers \( k, \ell \geq 0 \),
\[
\sup_{x \in \mathbb{R}} |x|^k |f^{(\ell)}(x)| < \infty.
\]

It is not hard to see that this class of function is closed under the following operations and corresponding properties for the Fourier transform

- Translation: For \( h > 0 \), \( \tau_h f(x) = f(x - h) \),
\[
F(\tau_h f)(\xi) = e^{-2\pi i h \xi} \hat{f}(\xi);
\]
• Frequency shift: For $\eta > 0$, $f(x) \mapsto e^{2\pi i \eta x} f(x)$,
  \[ \mathcal{F}(e^{2\pi i \eta x} f(x))(\xi) = \hat{f}(\xi - \eta); \]

• Differentiation: $f(x) \mapsto f'(x)$,
  \[ \mathcal{F}(f')(\xi) = 2\pi i \xi \hat{f}(\xi); \quad (1.19) \]

• Multiplication by $x$: $f(x) \mapsto xf(x)$,
  \[ \mathcal{F}(xf(x))(\xi) = -\frac{1}{2\pi i} \frac{d}{d\xi} \hat{f}(\xi); \]

• Dilation: For $\delta > 0$, $f(x) \mapsto f(\delta x)$,
  \[ \mathcal{F}(f(\delta x))(\xi) = \delta^{-1} f(\delta^{-1} \xi); \]

• Multiplication: For $f, g \in \mathcal{S}$, $fg \in \mathcal{S}$,
  \[ \mathcal{F}(fg)(\xi) = \hat{f} \ast \hat{g}(\xi); \]

• Convolution: For $f, g \in \mathcal{S}$, $f \ast g(x) = \int f(y)g(x-y)dy$ is in $\mathcal{S}$,
  \[ \mathcal{F}(f \ast g)(\xi) = \hat{f}(\xi)\hat{g}(\xi). \]

In particular, the properties for differentiation and multiplication by $x$ actually show that for $f \in \mathcal{S}(\mathbb{R})$, $\hat{f}$ is also in $\mathcal{S}(\mathbb{R})$. Moreover, the inverse Fourier transform is actually given by the formula

\[ \mathcal{F}^{-1}g(x) = \int_{-\infty}^{\infty} e^{2\pi i x \xi} g(\xi) d\xi. \]

Again, if we denote by $\mathcal{R}$ the reversal operator: $\mathcal{R}f(x) = f(-x)$, then

\[ \mathcal{F}^{-1} = \mathcal{F} \mathcal{R} = \mathcal{R} \mathcal{F} \]

\[ \mathcal{F}^2 = \mathcal{R}, \quad \mathcal{F}^4 = I. \]

The Plancherel theorem now reads as follows: for any $f, g \in \mathcal{S}(\mathbb{R})$,

\[ \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi. \]

In particular, setting $f = g$, we find the analogue of the Parseval identity for Fourier series:

\[ \int |f(x)|^2 dx = \int |\hat{f}(\xi)|^2 d\xi. \]
In quantum mechanics, the uncertainty principle is first formulated by Heisenberg in 1927, which states that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa. Mathematically, this can be formulated in terms of a function and its Fourier transform. Roughly speaking, an uncertainty principle in harmonic analysis is a statement that a function and its Fourier transform can not be simultaneously localized (or be “small”).

2.1. **Heisenberg uncertainty principle.** The formal statement of the Heisenberg uncertainty principle can be formulated using the variance (or standard deviation) of a function and its Fourier transform. More precisely, we have the following inequality.

**Theorem 2.1.** If \( f \in \mathcal{S}(\mathbb{R}) \) and \( \int |f(x)|^2 dx = 1 \), then

\[
\left( \int |x|^2 |f(x)|^2 dx \right) \left( \int |\xi|^2 |\hat{f}(\xi)|^2 d\xi \right) \geq \frac{1}{16\pi^2}.
\]

Moreover, the equality holds if and only if \( \psi(x) = \sqrt{2a/\pi}e^{-ax^2} \) for some \( a > 0 \).

**Proof.** An integration by parts shows that

\[
1 = \int_{-\infty}^{\infty} |f(x)|^2 dx = -\int_{-\infty}^{\infty} x \frac{d}{dx} |f(x)|^2 dx = -2 \int_{-\infty}^{\infty} \text{Re}(xf(x)f'(x)) dx.
\]

By Cauchy-Schwarz inequality,

\[
1 \leq 2 \int_{-\infty}^{\infty} |x||f(x)||f'(x)| dx \leq 2 \left( \int |x|^2 |f(x)|^2 dx \right)^{1/2} \left( \int |f'(x)|^2 dx \right)^{1/2}.
\]

Finally to finish the proof of (2.1), we use that Fourier transform is unitary and (1.42),

\[
\int_{-\infty}^{\infty} |f'(x)|^2 dx = 4\pi^2 \int_{-\infty}^{\infty} \xi^2 |\hat{\psi}(\xi)|^2 d\xi.
\]

The equality holds if and only if \( f'(x) = cx f(x) \) for some real constant \( c \) which gives \( f(x) = Ae^{cx^2/2} \). Since \( f \) is Schwartz, \( c \) is negative and taking \( a = -c/2 \), we can compute \( A \) by the normalization \( \|f\| = 1 \).

**Remark 2.2.** Heisenberg’s uncertainty principle is one of the fundamental principle in quantum mechanics. In (2.1), \( |f|^2 \) represents the probability distribution of the position while \( |\hat{f}|^2 \) represents the probability distribution of the momentum. Thus the integrals represents the variation/standard deviation of the position/momentum.
2.2. Entropic uncertainty principles. Another important concept measuring the concentration of a function is the entropy, defined as in thermodynamics or information theory. Let \( f \in S(\mathbb{R}) \) and \( \int |f|^2 dx = 1 \), then \( |f|^2 \) is the density of a probability measure on \( \mathbb{R} \). The Shannon entropy of \( |f|^2 \) is defined by
\[
H(|f|^2) = -\int_{-\infty}^{\infty} |f(x)|^2 \log |f(x)|^2 dx. \tag{2.2}
\]
The following entropic uncertainty principle was conjectured by Hirschman and Everett, proved by Beckner \cite{Bec}.

**Theorem 2.3.** Let \( f \in S(\mathbb{R}) \) and \( \int |f|^2 dx = 1 \), then
\[
H(|f|^2) + H(|\hat{f}|^2) \geq \log(e/2).
\]
The equality holds if and only if \( f \) is Gaussian.

We also have the following discrete analogue of the entropic uncertainty principle. For \( u \in \mathbb{C}^N \) with \( \|u\| = 1 \), we define the entropy of the probabilistic distribution \( |u_j|^2 \) to be
\[
H(|u|^2) = -\sum_{j=0}^{N-1} |u_j|^2 \log |u_j|^2.
\]

**Theorem 2.4.** Let \( u \in \mathbb{C}^N \), \( \|u\| = 1 \) and let \( v = N^{-1/2} \mathcal{F}_N u = \mathcal{U}_N u \), so \( \|v\| = 1 \). Then
\[
H(|u|^2) + H(|v|^2) \geq \log N.
\]
The equality holds if and only if \( f \) is a suitable normalized Kronecker comb.

2.3. Deterministic uncertainty principles. Another way to analyze the localization of a function is to consider its support \( \text{supp} f = \{ f(x) \neq 0 \} \).

**Theorem 2.5.** Let \( f \in L^1(\mathbb{R}) \). If \( \text{supp} f \) and \( \text{supp} \hat{f} \) are both compact, i.e. contained in a finite interval, then \( f \equiv 0 \).

There is a stronger version due to Benedicks \cite{Ben}.

**Theorem 2.6.** Let \( f \in L^1(\mathbb{R}) \). If \( \{ f \neq 0 \} \) and \( \{ \hat{f} \neq 0 \} \) both have finite measure, then \( f \equiv 0 \).

For discrete Fourier transform, we have the following results which is useful in signal recovery, see Donoho–Stark \cite{DoSt}.

**Theorem 2.7.** Let \( u \in \ell^2(\mathbb{Z}_N) \) and \( u \neq 0 \). Let \( X = \{ j \in \mathbb{Z}_N : u_j \neq 0 \} \), \( Y = \{ k \in \mathbb{Z}_N : \hat{u}_k \neq 0 \} \), then \( \#(X) \cdot \#(Y) \geq N \). The equality holds if and only if \( X \) and \( Y \) are both arithmetic progression and \( u \) is given by a multiple of the indicator function on \( X \) (Kronecker comb) or its shift in frequency domain.
Theorem 2.8. Let $u \in \ell^2(\mathbb{Z}_N)$ and $u \neq 0$. Let $X = \{ j \in \mathbb{Z}_N : u_j \neq 0 \}$, $Y = \{ k \in \mathbb{Z}_N : \hat{u}_k \neq 0 \}$. If $X$ has a gap of size $L$, i.e. there are $L$ consecutive elements in $\mathbb{Z}_N$ missing from $X$, then $|Y| > N - L$.

3. Exercises

1. Prove the properties of the discrete Fourier transform and the continuous Fourier transform.

2. Show that $F_N$ diagonalizes all circulant matrices and find the resulting diagonal matrices, i.e. the eigenvalues.

3. Write down the fast Fourier transform algorithm for $N = p^n$ for any $p \geq 3$ and check that it indeed gives the discrete Fourier transform in $O(N \log N)$ operations.

References


E-mail address: kdatech@purdue.edu

E-mail address: long249@purdue.edu

Department of Mathematics, Purdue University, 150 N. University St, West Lafayette, IN 47907