MA 428 final review problems

Finalized version as of April 27th

The final exam will be in STON 217 from 7 pm to 9 pm on Tuesday, May 4th. No notes or electronic devices are allowed. The problems on the exam will be closely based on parts of problems from the list below. For each problem, you must justify your answers. Please let me know if you have a question or find a mistake.

1. Let a and b be given real numbers, and let

$$f_n(x) = \frac{n^a}{n^b + x^2}$$

For which values of a and b do we have each of the following? Sketch the corresponding regions in the (a,b) plane.

(a)
$$\lim_{n\to\infty} f_n(x) = 0 \text{ for almost every } x \in \mathbb{R}.$$

(b)
$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_n(x)| = 0.$$

(c)
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n| = 0.$$

(d)
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n|^2 = 0.$$

2. Let $a > 0$ be given, and let

$$f(x) = \begin{cases} \sin(x^2), & 0 \le x \le a, \\ 0, & \text{otherwise,} \end{cases} \qquad \hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\xi} f(t)\,dt.$$

(a) Find a number b, depending on a, such that

$$|\hat{f}(\xi)| \le b/|\xi|,$$

for all $\xi \neq 0$.

(b) For which values of a can you find a number c, depending on a, such that

$$|\hat{f}(\xi)| \le c/|\xi|^2,$$

for all $\xi \neq 0$?

3. Find a constant a, as large as possible, and a constant b, as small as possible, such that

$$a \le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\exp(-t^4)}{(x-t)^2 + y^2}dt \le b.$$

for all $x \in \mathbb{R}$ and $y > 0$. You may use the fact that

$$\int_{-\infty}^{\infty} \frac{y}{s^2 + y^2} ds = \pi,$$

for all $y > 0$. Can you show that the constants you found are optimal?

4. For $a > 0$, let

$$f_a(x) = \begin{cases} 1, & |x| \leq a, \\ 0, & \text{otherwise}, \end{cases} \qquad \hat{f}_a(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\xi} f_a(t) dt.$$

   Use Plancherel's formula

$$\int_{-\infty}^{\infty} f\bar{g} = 2\pi \int_{-\infty}^{\infty} \hat{f}\bar{\hat{g}}$$

   to evaluate

$$\int_0^{\infty} \frac{\sin(bx)\sin(cx)}{x^2} dx,$$

   for any given $b > 0$ and $c > 0$.

5. Find constants a, b, and c which minimize the error

$$E = \int_{-\pi}^{\pi} |f(\theta) - a - b\cos\theta - c\sin(\theta/5)|^2 d\theta,$$

   where

$$f(\theta) = \begin{cases} 2, & 0 \leq \theta \leq \pi \\ -1, & -\pi < \theta < 0. \end{cases}$$

   You do not have to compute the error E.
   Hint: $2\cos^2 x = 1 + \cos 2x$ and $2\sin^2 x = 1 - \cos 2x$.

6. Let a and b be given real numbers such that $-\pi < a < b < \pi$, and let $f(\theta)$ be the 2π-periodic function given by

$$f(\theta) = \begin{cases} 0, & -\pi \leq \theta \leq a \\ 1, & a < \theta < b \\ 0, & b \leq \theta \leq \pi. \end{cases}$$

   The solution to the heated ring problem $\partial_t u(t,\theta) = \partial_\theta^2 u(t,\theta)$ with $u(0,\theta) = f(\theta)$ is given by

$$u(t,\theta) = \sum_{n=-\infty}^{\infty} c_n e^{-n^2 t} e^{in\theta}, \qquad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

   Find the equilibrium temperature $\lim_{t\to\infty} u(t,\theta)$. Find a time T, independent of a and b, such that $u(t,\theta)$ is within 1% of the equilibrium temperature for all $t \geq T$, regardless of a and b.

7. For which values of p is $\sum_{-\infty}^{\infty} \frac{n}{|n|^p+1} e^{in\theta}$ the Fourier series of a function in L^2?

8. Use the Fourier transform you found in problem 4 to find the Fourier transform of $f(x) = \sin(cx)/x$, where $c > 0$.

9. Let $f \in L^1$ be continuous and such that

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\xi} f(t) dt = \frac{1}{\xi^2 + e^{\xi} + 1}.$$

Recall that then

$$f(x) = \int_{-\infty}^{\infty} e^{ix\xi} \hat{f}(\xi) d\xi.$$

Let $g(x) = 2f(3x-4)$. Find $\hat{g}(\xi)$. Find $\int_{-\infty}^{\infty} g$. Find a number a such that $|g(x)| \leq a$ for all x.

10. The absolute convergence theorem says that if $g_1, g_2, \ldots$ are integrable functions, and $\sum \int |g_n|$ converges, then $\sum g_n$ is integrable and $\int \sum g_n = \sum \int g_n$. Use this and the power series $e^s = \sum_{n=0} s^n/n!$ to show that, for every real b, there are coefficients a_n depending on a and b (written in terms of an integral that you don't need to try to explicitly evaluate) such that

$$\hat{f}(\xi) = \sum_{n=0}^{\infty} a_n (\xi - b)^n,$$

for all ξ, where $\hat{f}$ is as in problem 2. (In other words, $\hat{f}$ has a convergent power series expansion about every point. It follows that $\hat{f}$, in contrast with f, is infinitely differentiable at all points and has only isolated zeroes.)