XII.

On the representation of a function by a trigonometric series.

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The following essay on trigonometric series consists of two essentially different parts. The first part contains a history of the research and opinions on arbitrary (graphically given) functions and their representation by trigonometric series. In its composition I was guided by some hints of the famous mathematician, to whom the first fundamental work on this topic was due. In the second part, I examine the representation of a function by a trigonometric series including cases that were previously unresolved. For this, it was necessary to start with a short essay on the concept of a definite integral and the scope of its validity.

History of the question of the representation of an arbitrary function by a trigonometric series.

1.

The trigonometric series named after Fourier, that is, the series of the form

$$\begin{aligned} &a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \\ +\frac{1}{2}\, &b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots \end{aligned}$$

play a significant role in those parts of mathematics where arbitrary functions occur. Indeed, there is reason to assert that the most substantial progress in this part of mathematics, that is so important for physics, has depended on a clear insight into the nature of these series. As soon as mathematical research first led to consideration of arbitrary functions, the question arose whether an arbitrary function could be expressed by a series of the above form.

This occurred in the middle of the eighteenth century during the study of vibrating strings, a topic in which the most prominent mathematicians of the time were interested. Their insights about our topic would probably not be represented were it not for the investigation of this problem.

As is well known, under certain hypotheses that conform approximately to reality, the shape of a string under tension that is vibrating in a plane is determined by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

where x is the distance of an arbitrary one of its points from the origin and y is the distance from the rest position at time t. Furthermore α is independent of t, and also of x for a string of uniform thickness.

D'Alembert was the first to give a general solution to this differential equation.

He showed[1] that each function of x and t, which when set in the equation for y yields an identity, must have the form

$$f(x+\alpha t) + \phi(x-\alpha t).$$

This follows by introducing the independent variables $x+\alpha t$, $x-\alpha t$ instead of x and t, whereby

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{\alpha^2}\frac{\partial^2 y}{\partial t^2} \quad \text{changes into} \quad 4\,\frac{\partial \frac{\partial y}{\partial(x+\alpha t)}}{\partial(x-\alpha t)}.$$

Besides the partial differential equation, which results from the general laws of motion, y must also satisfy the condition that it is always 0 at the endpoints of the string. Thus, if one of these points is at $x=0$ and the other at $x=\ell$, we have

$$f(\alpha t) = -\phi(-\alpha t), \qquad f(\ell+\alpha t) = -\phi(\ell-\alpha t)$$

and consequently

$$f(z) = -\phi(-z) = -\phi(\ell-(\ell+z)) = f(2\ell+z),$$
$$y = f(\alpha t + x) - f(\alpha t - x).$$

After d'Alembert had succeeded in finding the above for the general solution of the problem, he treated, in a sequel[2] to his paper, the equation

[1] *Mémoires de l'académie de Berlin*, 1747, p. 214.
[2] Ibid. p. 220.

$f(z) = f(2\ell + z)$. That is, he looked for analytic expressions that remained unchanged if z is increased by 2ℓ.

In the next issue of *Mémoires de l'académie de Berlin*[3], Euler made a basic advance, giving a new presentation of d'Alembert's work and recognizing more exactly the nature of the conditions which the function $f(x)$ must satisfy. He noted that, by the nature of the problem, the movement of the string is completely determined, if at some point in time the shape of the string and the velocity are given at each point (that is, y and $\frac{\partial y}{\partial t}$). He showed that if one thinks of the two functions as being determined by arbitrarily drawn curves, then the d'Alembert function $f(z)$ can always be found by a simple geometric construction. In fact, if one assumes that $y = g(x)$ and $\frac{\partial y}{\partial t} = h(x)$ when $t = 0$, then one obtains

$$f(x) - f(-x) = g(x) \quad \text{and} \quad f(x) + f(-x) = \frac{1}{\alpha}\int h(x)\,dx$$

for values of x between 0 and ℓ, and hence obtains the function $f(z)$ between $-\ell$ and ℓ. From this, however, the values of $f(z)$ can be derived for all other values of z using the equation

$$f(z) = f(2\ell + z).$$

This is, represented in abstract but now generally accepted concepts, Euler's determination of the function $f(z)$.

D'Alembert at once protested against this extension of his methods by Euler[4], since it assumed that y could be expressed analytically in t and x.

Before Euler replied to this, Daniel Bernoulli[5] presented a third treatment of this topic, which was quite different from the previous two. Even prior to d'Alembert, Taylor[6] had seen that $y = \sin\frac{n\pi x}{\ell}\cos\frac{n\pi\alpha t}{\ell}$, where n is an integer, satisfies $\frac{\partial^2 y}{\partial t^2} = \alpha^2\frac{\partial^2 y}{\partial x^2}$ and always equals 0 for $x = 0$ and $x = \ell$. From this he explained the physical fact that a string, besides its fundamental tone, can also give the fundamental tone of a string that is $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ as

[3] *Mémoires de l'académie de Berlin*, 1748, p. 69.

[4] *Mémoires de l'académie de Berlin*, 1750, p. 358. 'Indeed, it seems to me, one can only express y analytically in a more general fashion by supposing it is a function of t and x. But with this assumption one only finds a solution of the problem for the case where the different graphs of the vibrating string can be contained in a single equation.'

[5] *Mémoires de l'academie de Berlin*, 1753, p. 147.

[6] Taylor, *De methode incrementorum*.

long (but otherwise similarly constituted). He took his particular solutions as general: he thought that if the pitch of the tone was determined by the integer n, then the vibration of the string would always be as expressed by the equation, or at least very nearly. The observation that a string could simultaneously sound different notes now led Bernoulli to the remark that the string (by the theory) could also vibrate in accordance with the equation

$$y = \sum a_n \sin \frac{n\pi x}{\ell} \cos \frac{n\pi\alpha}{\ell} (t - \beta_n).$$

Further, since all observed modifications of the phenomenon could be explained by this equation, he considered it the most general solution.[7] In order to support this opinion, he examined the vibration of a massless thread under tension, which was weighted at isolated points with finite masses. He showed that the vibrations can be decomposed into a number of vibrations that is always equal to the number of points, each vibration being of the same duration for all masses.

This work of Bernoulli prompted a new paper from Euler, which was printed immediately following it in the *Mémoires de l'académie de Berlin.*[8] He maintained, in opposition to d'Alembert[9], that the function $f(z)$ could be completely arbitrary between $-\ell$ and ℓ. Euler[10] noted that Bernoulli's solution (which he had previously represented as particular) is general if and only if the series

$$\begin{aligned} &a_1 \sin \frac{x\pi}{\ell} + a_2 \sin \frac{2x\pi}{\ell} + \cdots \\ +&\frac{1}{2} b_0 + b_1 \cos \frac{x\pi}{\ell} + b_2 \cos \frac{2x\pi}{\ell} + \cdots \end{aligned}$$

can represent the ordinate of an arbitrary curve for the abcissa x between 0 and ℓ. Now no one doubted at that time that all transformations which could be made with an analytic expression (finite or infinite) would be valid for each value of the variable, or only inapplicable in very special cases. Thus it seemed impossible to represent an algebraic curve, or in general a nonperiodic analytically given curve, by the above expression. Hence Euler thought that the question must be decided against Bernoulli.

[7]Loc. cit., p. 157 section XIII.

[8]*Mémoires de l'académie de Berlin*, 1753, p. 196.

[9]Loc. cit., p. 214

[10]Loc. cit., sections III–X.

The disagreement between Euler and d'Alembert was still unresolved by this. This induced the young, and then little known, mathematician Lagrange to seek the solution of the problem in a completely new way, by which he reached Euler's results. He undertook[11] to determine the vibration of a massless thread which is weighted with an indeterminate finite number of equal masses that are equally spaced. He then examined how the vibrations change when the number of masses grows towards infinity. Although he carried out the first part of this investigation with much dexterity and a great display of analytic ingenuity, the transition from the finite to the infinite left much to be desired. Hence d'Alembert could continue to vindicate the reputation of his solution as the most general by making this point in a note in his *Opuscules Mathématiques.* The opinions of the prominent mathematicians of this time were, and remained, divided on the matter; for in later work everyone essentially retained his own point of view.

In order to finally arrange his views on the problem of arbitrary functions and their representation by trigonometric series, Euler first introduced these functions into analysis, and supported by geometrical considerations, applied infinitesimal analysis to them. Lagrange[12] considered Euler's results (his geometric construction for the course of the vibration) to be correct, but he was not satisfied with Euler's geometric treatment of the functions. D'Alembert,[13] on the other hand, acceded to Euler's way of obtaining the differential equation and restricted himself to disputing the validity of his result, since one could not know for an arbitrary function whether its derivatives were continuous. Concerning Bernoulli's solution, all three agreed not to consider it as general. While d'Alembert,[14] in order to explain Bernoulli's solution as less general than his own, had to assert that an analytically given periodic function cannot always be represented by a trigonometric series, Lagrange[15] believed it possible to prove this.

2.

Almost fifty years had passed without a basic advance having been made in the question of the analytic representation of an arbitrary function. Then

[11] *Miscellanea Taurinensia*, vol. I. Recherches sur la nature et la propagation du son.
[12] *Miscellanea Taurinensia*, vol. II, *Pars math.*, p. 18.
[13] *Opuscules Mathématiques*, d'Alembert. Vol. 1, 1761, p. 16, Sections VII—XX.
[14] *Opuscules Mathématiques*, vol. I, p. 42, Section XXIV.
[15] *Misc. Taur. vol. III, Pars math.*, p. 221, Section XXV.

a remark by Fourier threw a new light on the topic. A new epoch in the development of this part of mathematics began, which soon made itself known in a wonderful expansion of mathematical physics. Fourier noted that in the trigonometric series

$$f(x) = \begin{cases} a_1 \sin x + a_2 \sin 2x + \cdots \\ +\frac{1}{2}\, b_0 + b_1 \cos x + b_2 \cos 2x + \cdots , \end{cases}$$

the coefficients can be determined by the formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx.$$

He saw that the method can also be applied if the function $f(x)$ is arbitrary. He used a so-called discontinuous function for $f(x)$ (with ordinate a broken line for the abscissa x) and obtained a series which in fact always gives the value of the function.

Fourier, in one of his first papers on heat, which was submitted to the French academy[16] (December 21, 1807) first announced the theorem, that an arbitrary (graphically given) function can be expressed as a trigonometric series. This claim was so unexpected to the aged Lagrange that he opposed it vigorously. There should[17] be another note about this in the archives of the Paris academy. Nevertheless, Poisson refers,[18] whenever he makes use of trigonometric series to represent arbitrary functions, to a place in Lagrange's work on the vibrating string where this method of representation can be found. In order to refute this claim, which can only be explained by the well known rivalry[19] between Fourier and Poisson, we must once again return to Lagrange's treatise, since nothing can be found that is published about these facts by the academy.

In fact, one finds in the place cited[20] by Poisson the formula:

$$`y = 2\int Y \sin X\pi \, dX \times \sin x\pi + 2\int Y \sin 2X\pi \, dX \times \sin 2x\pi$$
$$+ 2\int Y \sin 3X\pi \, dX \times \sin 3x\pi + \text{etc.} + 2\int Y \sin nX\pi \, dX \times \sin nx\pi,$$

[16] *Bulletin des sciences p. la soc. philomatique*, vol I, p. 112.

[17] From a verbal report of Professor Dirichlet.

[18] Among others, in the expanded *Traité de mécanique* No. 323, p. 638

[19] The review in the *Bulletin des Sciences* on the paper submitted by Fourier to the academy was written by Poisson.

[20] *Misc. Taur.*, vol. III, *Pars math.*, p. 261.

so that when $x = X$, one has $y = Y$, Y being the ordinate corresponding to the abscissa X'.

This formula looks so much like a Fourier series that is easy to confuse them with just a quick glance. However, this appearance arises only because Lagrange uses $\int dX$ where today we would use $\sum \Delta X$. It gives the solution to the problem of determining the finite sine series

$$a_1 \sin x\pi + a_2 \sin 2x\pi + \cdots + a_n \sin nx\pi$$

so that it has given values when x equals

$$\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}.$$

Lagrange denotes the variable by X. If Lagrange had let n become infinitely large in this formula, then certainly he would have obtained Fourier's result. However, if we read through his paper, we see that he was far from believing that an arbitrary function could actually be represented by an infinite sine series. Rather, he had undertaken the whole work because be believed that an arbitrary function could not be expressed by a formula. Concerning trigonometric series, he thought they could be used to represent any analytically given periodic function. Admittedly, it now seems scarcely possible that Lagrange did not obtain Fourier's series from his summation formula. However, this can be explained in that the dispute between Euler and d'Alembert had predisposed him towards a particular opinion about the proper method of proceeding. He thought that the vibration problem, for an indeterminate finite number of masses, must be fully solved before applying limit considerations. This necessitated a rather extensive investigation[21], which was unnecessary if he had been acquainted with the Fourier series.

The nature of the trigonometric series was recognized perfectly correctly by Fourier.[22] Since then these series have been applied many times in mathematical physics to represent arbitrary functions. In each individual case it was easy to convince oneself that the Fourier series really converged to the value of the function. However, it was a long time before this important theorem would be proved in general.

[21] *Misc. Taur.*, vol III, *Pars math.*, p. 251.

[22] *Bulletin d. sc.* vol. I, p. 115. 'The coefficients $a, a', a'', \ldots$, being then determined', etc.

The proof which Cauchy[23] read to the Paris academy on February 27, 1826, is inadequate, as Dirichlet[24] has shown. Cauchy assumed that if x is replaced by the complex argument $x + yi$ in an arbitrary periodic function $f(x)$, then the function is finite for each value of y. However, this only occurs if the function is a constant. It is easy to see that this hypothesis was unnecessary for the later conclusions. It suffices that a function $\phi(x + yi)$ exists which is finite for all positive values of y, whose real part is equal to the given periodic function $f(x)$ when $y = 0$. If one assumes this theorem, which is in fact true,[25] then Cauchy's method certainly leads to the goal; conversely, this theorem can be derived from the Fourier series.

3.

The question of the representation by trigonometric series of everywhere integrable functions with finitely many maxima and minima was first settled rigorously by Dirichlet[26] in a paper of January 1829.

The recognition of the proper way to attack the problem came to him from the insight that infinite series fall into two distinct classes, depending on whether or not they remain convergent when all the terms are made positive. In the first class the terms can be arbitrarily rearranged; in the second, on the other hand, the value is dependent on the ordering of the terms. Indeed, if we denote the positive terms of a series in the second class by

$$a_1, a_2, a_3, \ldots,$$

and the negative terms by

$$-b_1,\ -b_2,\ -b_3, \ldots,$$

then it is clear that $\sum a$ as well as $\sum b$ must be infinite. For if they were both finite, the series would still be convergent after making all the signs the same. If only one were infinite, then the series would diverge. Clearly now an arbitrarily given value C can be obtained by a suitable reordering of the terms. We take alternately the positive terms of the series until the sum is greater than C, and then the negative terms until the sum is less than C. The deviation from C never amounts to more than the size of the term at

[23] *Mémoires de l'ac. d. sc. de Paris*, vol. VI, p. 603.
[24] *Crelle's Journal für die Mathematik*, vol IV, pp. 157 & 158.
[25] The proof can be found in the inaugural dissertation of the author.
[26] *Crelle's Journal*, vol. IV, p. 157.

the last place the signs were switched. Now, since the numbers a as well as the numbers b become infinitely small with increasing index, so also are the deviations from C. If we proceed sufficiently far in the series, the deviation becomes arbitrarily small, that is, the series converges to C.

The rules for finite sums only apply to the series of the first class. Only these can be considered as the aggregates of their terms; the series of the second class cannot. This circumstance was overlooked by mathematicians of the previous century, most likely, mainly on the grounds that the series which progress by increasing powers of a variable generally (that is, excluding individual values of this variable) belong to the first class.

Clearly the Fourier series do not necessarily belong to the first class. The convergence cannot be derived, as Cauchy futilely attempted,[27] from the rules by which the terms decrease. Rather, it must be shown that the finite series

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin\alpha\, d\alpha\, \sin x + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin 2\alpha\, d\alpha \sin 2x + \cdots$$
$$+\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin n\alpha\, d\alpha\, \sin nx$$
$$+\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\, d\alpha + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos\alpha\, d\alpha \cos x$$
$$+\frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos 2\alpha\, d\alpha\, \cos 2x + \cdots + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos n\alpha\, d\alpha \cos nx,$$

or, what is the same, the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\ \frac{\sin\frac{2n+1}{2}(x-\alpha)}{\sin\frac{x-\alpha}{2}}\ d\alpha,$$

approaches the value $f(x)$ infinitely closely when n increases infinitely.

Dirichlet based this proof on two theorems:

1) If $0 < c \le \pi/2$, then $\int_0^c \phi(\beta)\frac{\sin(2n+1)\beta}{\sin\beta}\, d\beta$ tends to $\frac{\pi}{2}\,\phi(0)$ as n increases to infinity.
2) If $0 < b < c \le \pi/2$, then $\int_b^c \phi(\beta)\frac{\sin(2n+1)\beta}{\sin\beta} d\beta$ tends to 0, as n increases to infinity.

[27]Dirichlet in *Crelle's Journal*, vol IV, p. 158. 'Quoi qu'il en soit de cette première observation, ...à mesure que n croit.'

It is assumed in both cases that the function $\phi(\beta)$ is either always increasing or always decreasing between the limits of integration.

If the function f does not change from increasing to decreasing, or from decreasing to increasing, infinitely often, then using the above theorems the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\,\frac{\sin\frac{2n+1}{2}(x-\alpha)}{\sin\frac{x-\alpha}{2}}\,d\alpha$$

can clearly be split into a finite number of parts, one of which tends[28] to $\frac{1}{2}f(x+0)$, another to $\frac{1}{2}f(x-0)$, and the others to 0, as n increases to infinity.

It follows from this that a periodic function of period 2π, which

1. is everywhere integrable,

2. does not have infinitely many maxima and minima, and

3. assumes the average of the two one-sided limits when the value changes by a jump,

can be represented by a trigonometric series.

It is clear that a function satisfying the first two properties but not the third cannot be represented by a trigonometric series. A trigonometric series representing such a function, except at the discontinuities, would deviate from it at the discontinuities. Dirichlet's research leaves undecided, whether and when functions can be represented by a trigonometric series that do not satisfy the first two conditions.

Dirichlet's work gave a firm foundation for a large amount of important research in analysis. He succeeded in bringing light to a point where Euler was in error. He settled a question that had occupied many distinguished mathematicians for over 70 years (since 1753). In fact, for all cases of nature, the only cases concerned in that work, it was completely settled. For however great our ignorance about how forces and states of matter vary for infinitely small changes of position and time, surely we may assume that the functions which are not included in Dirichlet's investigations do not occur in nature.

[28]It is easy to prove that the value of a function f, which does not have infinitely many maxima or minima, for increasing or decreasing values of the argument with limit x, either approaches fixed limits $f(x+0)$ and $f(x-0)$ (using Dirichlet's notation in Dove's *Repertorium der Physik*, vol. 1, p. 170); or must become infinitely large.

Nevertheless, there are two reasons why those cases unresolved by Dirichlet seem to be worthy of consideration.

First, as Dirichlet noted at the end of his paper, the topic has a very close connection with the principles of infinitesimal calculus, and can serve to bring greater clarity and rigor to these principles. In this regard the treatment of the topic has an immediate interest.

Secondly, however, the applications of Fourier series are not restricted to research in the physical sciences. They are now also applied with success in an area of pure mathematics, number theory. Here it is precisely the functions whose representation by a trigonometric series was not examined by Dirichlet that seem to be important.

Admittedly Dirichlet promised at the conclusion of his paper to return to these cases later, but that promise still remains unfulfilled. The works by Dirksen and Bessel on the cosine and sine series did not supply this completion. Rather, they take second place to Dirichlet in rigor and generality. Dirksen's paper,[29] (almost simultaneous with Dirichlet's, and clearly written without knowledge of it) was, indeed, in a general way correct. However, in the particulars it contained some imprecisions. Apart from the fact that he found an incorrect result in a special case[30] for the sum of a series, he relied in a secondary consideration on a series expansion[31] that is only possible in particular cases. Hence the proof is only complete for functions whose first derivatives are everywhere finite. Bessel[32] tried to simplify Dirichlet's proof. However, the changes in the proof did not give any essential simplification, but at most clothed it in more familiar concepts, at the expense of rigor and generality.

Hence, until now, the question of the representation of a function by a trigonometric series is only settled under the two hypotheses, that the function is everywhere integrable and does not have infinitely many maxima and minima. If the last hypothesis is not made, then the two integral theorems of Dirichlet are not sufficient for deciding the question. If the first is discarded, however, the Fourier method of determining the coefficients is not applicable. In the following, when we examine the question without any particular assumptions on the nature of the function, the method employed, as we will see, is constrained by these facts. An approach as direct as Dirichlet's is not

[29] *Crelle's Journal*, vol IX, p. 170.
[30] *Loc. cit.*, formula 22.
[31] *Loc. cit.*, section 3.
[32] Schumacher, *Astronomische Nachrichten*, 374 (vol. 16, p. 229.)

possible by the nature of the case.

On the concept of a definite integral and the range of its validity.

4.

Vagueness still prevails in some fundamental points concerning the definite integral. Hence I provide some preliminaries about the concept of a definite integral and the scope of its validity.

Hence first: What is one to understand by $\int_a^b f(x)\,dx$?

In order to establish this, we take a sequence of values $x_1, x_2, \ldots, x_{n-1}$ between a and b arranged in succession, and denote, for brevity, $x_1 - a$ by δ_1, $x_2 - x_1$ by $\delta_2, \ldots, b - x_{n-1}$ by δ_n, and a positive fraction less than 1 by ϵ. Then the value of the sum

$$S = \delta_1 f(a + \epsilon_1\delta_1) + \delta_2 f(x_1 + \epsilon_2\delta_2) + \delta_3 f(x_2 + \epsilon_3\delta_3) + \cdots + \delta_n f(x_{n-1} + \epsilon_n\delta_n)$$

depends on the selection of the intervals δ and the numbers ϵ. If this now has the property, that however the δ's and ϵ's are selected, S approaches a fixed limit A when the δ's become infinitely small together, this limiting value is called $\int_a^b f(x)\,dx$.

If we do not have this property, then $\int_a^b f(x)\,dx$ is undefined. In some of these cases, attempts have been made to assign a meaning to the symbol, and among these extensions of the concept of a definite integral there is one recognized by all mathematicians. Namely, if the function $f(x)$ becomes infinitely large when the argument approaches an isolated value c in the interval (a, b), then clearly the sum S, no matter what degree of smallness one may prescribe for δ, can reach an arbitrarily given value. Thus it has no limiting value, and by the above $\int_a^b f(x)\,dx$ would have no meaning. However if

$$\int_a^{c-\alpha_1} f(x)\,dx + \int_{c+\alpha_2}^b f(x)\,dx$$

approaches a fixed limit when α_1 and α_2 become infinitely small, then one understands this limit to be $\int_a^b f(x)\,dx$.

Other hypotheses by Cauchy on the concept of the definite integral in the cases where the fundamental concepts do not give a value may be appropriate in individual classes of investigation. These are not generally established, and are hardly suited for general adoption in view of their great arbitrariness.

5.

Let us examine now, secondly, the range of validity of the concept, or the question: In which cases can a function be integrated, and in which cases can it not?

We consider first the concept of integral in the narrow sense, that is, we suppose that the sum S converges if the δ's together become infinitely small. We denote by D_1 the greatest fluctuation of the function between a and x_1, that is, the difference of its greatest and smallest values in this interval, by D_2 the greatest fluctuation between x_1 and $x_2, \ldots$, by D_n that between x_{n-1} and b. Then

$$\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$$

must become infinitely small when the δ's do. We suppose further, that Δ is the greatest value this sum can reach, as long as all of the δ's are smaller than d. Then Δ will be a function of d, which is decreasing with d and becomes infinitely small with d. Now, if the total length of the intervals, in which the fluctuation is greater than σ, is s, then the contribution of these intervals to the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ is clearly $\geq \sigma s$. Therefore one has

$$\sigma s \leq \delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n \leq \Delta, \text{ hence } s \leq \frac{\Delta}{\sigma}.$$

Now, if σ is given, Δ/σ can always be made arbitrarily small by a suitable choice of d. The same is true for s, which yields:

In order for the sum S to converge whenever all the δ's become infinitely small, in addition to $f(x)$ being finite, it is necessary that the total length of the intervals, in which the fluctuations exceed σ, can be made arbitrarily small for any given σ by a suitable choice of d.

This theorem also has a converse:

If the function $f(x)$ is always finite, and by infinitely decreasing the δ's together, the total length s of the intervals in which the fluctuation of the function is greater than a given number σ always becomes infinitely small, then the sum S converges as the δ's become infinitely small together.

For those intervals in which the fluctuations are $> \sigma$ make a contribution to the sum $\delta_1 D_1 + \cdots + \delta_n D_n$, less than s times the largest fluctuation of the function between a and b, which is finite (by agreement). The contribution of the remaining intervals is $< \sigma(b-a)$. Clearly one can now choose σ arbitrarily small and then always determine the size of the intervals (by agreement) so that s is also arbitrarily small. In this way the sum $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$

can be made as small as desired. Consequently the value of the sum S can be enclosed between arbitrarily narrow bounds.

Thus we have found necessary and sufficient conditions for the sum S to be convergent when the quantities δ tend together to zero, or equivalently, for the existence of the integral of $f(x)$ between a and b in the narrow sense.

If we now extend the integral concept as above, then it is clear that for the integration to be possible everywhere, the second of the two conditions established is still necessary. In place of the condition, that the function is always finite, will enter the condition, that the function becomes infinite only on the approach of the argument to isolated values, and that a definite limiting value arises, if the limits of integration tend to these values.

6.

Having examined the conditions for integrability in general, that is, without special assumptions on the nature of the function to be integrated, this investigation will be applied and also carried further, in special cases. First we consider functions which are discontinuous infinitely often between any two numbers, no matter how close.

Since these functions have never been considered before, it is well to start from a particular example. Designate, for brevity, (x) to be the excess of x over the closest integer, or if x lies in the middle between two (and thus the determination is ambiguous) the average of the two numbers $1/2$ and $-1/2$, hence zero. Furthermore, let n be an integer and p an odd integer, and form the series

$$f(x) = \frac{(x)}{1} + \frac{(2x)}{4} + \frac{(3x)}{9} + \cdots = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}.$$

It is easy to see that the series converges for each value of x. When the argument continuously decreases to x, as well as when it continuously increases to x, the value always approaches a fixed limit. Indeed, if $x = \frac{p}{2n}$ (where p and n are relatively prime)

$$f(x+0) = f(x) - \frac{1}{2n^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) = f(x) - \frac{\pi^2}{16n^2},$$

$$f(x-0) = f(x) + \frac{1}{2n^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) = f(x) + \frac{\pi^2}{16n^2};$$

in all other cases $f(x+0) = f(x)$ and $f(x-0) = f(x)$.

Hence this function is discontinuous for each rational value of x, which in lowest terms is a fraction with even denominator. Thus, while f is discontinuous infinitely often between any two bounds, the number of jumps greater than a fixed number is always finite. The function is everywhere integrable. Besides finiteness, it has the two properties, that for each value of x it has limiting values $f(x+0)$ and $f(x+0)$ on both sides, and that the number of jumps greater than or equal to a given number σ is always finite. Applying our above investigation, as an obvious consequence of these two conditions, d can be taken so small that in the intervals which do not contain these jumps, the fluctuations are smaller then σ, and the total length of the intervals which do contain these jumps will be arbitrarily small.

It is worthwhile to note that functions which do not have infinitely many maxima and minima (to which the example just considered does not belong), where they do not become infinite, always have those two properties, and hence permit an integration everywhere where they are not infinite. This is also easy to show directly.

Now consider the case where the function $f(x)$ to be integrated has a single infinite value. We assume this occurs at $x = 0$, so that for decreasing positive values of x its value eventually grows over any given bound.

It can easily be shown that $xf(x)$ cannot always remain larger then finite number c as x decreases from a finite bound a. For then we would have

$$\int_x^a f(x)\,dx > c\int_x^a \frac{dx}{x},$$

thus larger than $c\left(\log\frac{1}{x} - \log\frac{1}{a}\right)$, which increases to infinity with decreasing x. Thus if $xf(x)$ does not have infinitely many maxima and minima in a neighborhood of $x = 0$, then $xf(x)$ must become infinitely small with x if $f(x)$ can be integrated. On the other hand, if

$$f(x)x^\alpha = \frac{f(x)\,dx\,(1-\alpha)}{d(x^{1-\alpha})},$$

for a value $\alpha < 1$, becomes infinitely small with x, then it is clear that the integral converges as the lower limit tends to 0.

In the same way one finds that in the cases where the integral converges, the functions

$$f(x)x\log\frac{1}{x}=\frac{f(x)\,dx}{-d\log\log\frac{1}{x}},\quad f(x)x\log\frac{1}{x}\,\log\log\frac{1}{x}=\frac{f(x)\,dx}{-d\log\log\log\frac{1}{x}},\ldots,$$
$$f(x)x\log\frac{1}{x}\,\log\log\frac{1}{x}\cdots\log^{n-1}\frac{1}{x}\,\log^{n}\frac{1}{x}=\frac{f(x)\,dx}{-d\log^{1+n}\frac{1}{x}}$$

cannot remain always larger than a finite number as x decreases from a finite bound. Thus if they do not have infinitely many maxima and minima, these functions must become infinitely small with x. On the other hand, the integral $\int f(x)\,dx$ converges as the lower limit of integration tends to 0, if

$$f(x)x\log\frac{1}{x}\cdots\log^{n-1}\frac{1}{x}\left(\log^{n}\frac{1}{x}\right)^{\alpha}=\frac{f(x)\,dx\,(1-\alpha)}{-d\left(\log^{n}\frac{1}{x}\right)^{1-\alpha}}$$

becomes infinitely small with x, for $\alpha>1$.

However, if $f(x)$ has infinitely many maxima and minima, then nothing can be determined about the order at which it becomes infinite. In fact, given the absolute value of f, and thereby given the order of infinity of f at 0, by a suitable determination of the sign one can always make the integral $\int f(x)\,dx$ converge when the lower limit of integration tends to 0. The function

$$\frac{d\left(x\cos e^{1/x}\right)}{dx}=\cos e^{1/x}+\frac{1}{x}e^{1/x}\sin e^{1/x}$$

serves as an example of a function which becomes infinite in such a way that its order (taking the order of $\frac{1}{x}$ as one) is infinitely large.

The above discussion, on the principles of a topic belonging to another area, suffices. We now proceed to our actual problem, a general investigation of the representation of a function by a trigonometric series.

Investigation of the representation of a function by a trigonometric series without particular assumptions on the nature of the function.

7.

The previous work on this topic served the purpose of proving the Fourier series for the cases occurring in nature. Hence the proofs could start for an arbitrary function, and later for the purposes of the proof one could impose arbitrary restrictions on the function, when they did not impair the goal.

For our purposes we only impose conditions necessary for the representation of the function. Hence we must first look for necessary conditions for the representation, and from these select sufficient conditions for the representation. While the previous work showed: 'If a function has this or that property then it is represented by a Fourier series', we must start from the converse question: If a function is represented by a Fourier series, what are the consequences for the function, regarding the changes of its values with a continuous change of the argument?

Hence we consider the series

$$a_1 \sin x + a_2 \sin 2x + \cdots$$
$$+\frac{1}{2}\, b_0 + b_1 \cos x + b_2 \cos 2x + \cdots$$

as given. For brevity, set

$$\frac{1}{2}\, b_0 = A_0,\ \ a_1 \sin x + b_1 \cos x = A_1,\ \ a_2 \sin 2x + b_2 \cos 2x = A_2, \ldots;$$

the series becomes

$$A_0 + A_1 + A_2 + \cdots .$$

We denote this expression by Ω and its value by $f(x)$, so that this function is defined only for values of x where the series converges.

For the series to converge, it is necessary that the terms eventually become infinitely small. If the coefficients a_n and b_n diminish infinitely with increasing n, then the terms of the series Ω eventually become infinitely small for each value of x. Otherwise convergence can only occur for particular values of x. It is necessary to treat both cases separately.

8.

Hence we suppose, first of all, that the terms of the series Ω eventually become arbitrarily small for each x.

Under this assumption, the series

$$C + C'x + A_0 \frac{x^2}{2} - A_1 - \frac{A_2}{4} - \frac{A_3}{9} \ldots = F(x)$$

converges for each value of x. The series is obtained by integrating each term of Ω twice with respect to x. The value $F(x)$ changes continuously with x, and consequently this function F of x is everywhere integrable.

In order to establish both the convergence of the series and the continuity of $F(x)$, one denotes the sum of the terms to $-\frac{A_n}{n^2}$ inclusive by N, the remainder of the series, that is, the series

$$-\frac{A_{n+1}}{(n+1)^2} - \frac{A_{n+2}}{(n+2)^2} - \cdots$$

by R; and the greatest value of A_m for $m > n$ by ϵ. Then, no matter how far one continues the series, the absolute value of R clearly remains

$$< \epsilon \left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \right) < \frac{\epsilon}{n},$$

and R can be enclosed within arbitrarily small bounds if n is sufficiently large. Hence the series converges. Furthermore, the function $F(x)$ is continuous, that is, its variation can be made as small as we wish, if one imposes a sufficiently small corresponding change of x. For the combined changes of $F(x)$ consists of the change in R and in N. Clearly one can first assume that n is so large that R is arbitrarily small whatever x may be, and consequently also the change of R will be arbitrarily small for any change in x. Then assume the change of x is so small that the change in N also becomes arbitrarily small.

It is well to place here some results about the function $F(x)$, whose proofs would otherwise break the thread of the investigation.

Theorem 1 *If the series Ω converges, then*

$$\frac{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)}{4\alpha\beta},$$

converges to the same value as Ω when α and β become infinitely small while their ratio remains finite.

Indeed, we have

$$\frac{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)}{4\alpha\beta}$$
$$= A_0 + A_1 \frac{\sin\alpha}{\alpha}\frac{\sin\beta}{\beta} + A_2 \frac{\sin 2\alpha}{2\alpha}\frac{\sin 2\beta}{2\beta} + A_3 \frac{\sin 3\alpha}{3\alpha}\frac{\sin 3\beta}{3\beta} + \cdots .$$

In order to settle the simplest case $\beta = \alpha$ first,

$$\frac{F(x+2\alpha) - 2F(x) + F(x-2\alpha)}{4\alpha^2} = A_0 + A_1 \left(\frac{\sin\alpha}{\alpha}\right)^2 + A_2 \left(\frac{\sin 2\alpha}{2\alpha}\right)^2 + \cdots$$

If the infinite series converges,

$$A_0 + A_1 + A_2 + \ldots = f(x),$$

and we write

$$A_0 + A_1 + \cdots + A_{n-1} = f(x) + \epsilon_n,$$

then for an arbitrarily given number δ, there must exist an integer m so that if $n > m$ we have $\epsilon_n < \delta$. Now, assume α is so small that $m\alpha < \pi$. We use the substitution

$$A_n = \epsilon_{n+1} - \epsilon_n,$$

to put $\sum_{n=0}^{\infty} \left(\frac{\sin n\alpha}{n\alpha}\right)^2 A_n$ in the form

$$f(x) + \sum_{n=1}^{\infty} \epsilon_n \left\{ \left(\frac{\sin(n-1)\alpha}{(n-1)\alpha}\right)^2 - \left(\frac{\sin n\alpha}{n\alpha}\right)^2 \right\},$$

and separate this last infinite series into three parts, in which we put together

1. the terms of index 1 to m inclusive,

2. from index $m+1$ up to the largest integer s less than $\frac{\pi}{\alpha}$,

3. from $s+1$ to infinity.

Then the first part consists of a finite number of continuously varying terms, and therefore approaches its limiting value 0 arbitrarily closely when one lets α become sufficiently small. The second part, since the factor of ϵ_n is always positive, has absolute value

$$< \delta \left\{ \left(\frac{\sin m\alpha}{m\alpha}\right)^2 - \left(\frac{\sin s\alpha}{s\alpha}\right)^2 \right\}.$$

In order to enclose the third part within bounds, one breaks up the general term into

$$\epsilon_n \left\{ \left(\frac{\sin(n-1)\alpha}{(n-1)\alpha}\right)^2 - \left(\frac{\sin(n-1)\alpha}{n\alpha}\right)^2 \right\}$$

and

$$\epsilon_n \left\{ \left(\frac{\sin(n-1)\alpha}{n\alpha} \right)^2 - \left(\frac{\sin n\alpha}{n\alpha} \right)^2 \right\} = -\epsilon_n \frac{(\sin(2n-1)\alpha) \sin \alpha}{(n\alpha)^2}.$$

Hence clearly it is

$$< \delta \left\{ \frac{1}{(n-1)^2\alpha^2} - \frac{1}{n^2\alpha^2} \right\} + \delta \frac{1}{n^2\alpha}$$

and consequently the sum from $n = s + 1$ to ∞ is

$$< \delta \left\{ \frac{1}{(s\alpha)^2} + \frac{1}{s\alpha} \right\}.$$

For an infinitely small α, that number becomes

$$\delta \left\{ \frac{1}{\pi^2} + \frac{1}{\pi} \right\}.$$

The series

$$\sum \epsilon_n \left\{ \left(\frac{\sin(n-1)\alpha}{(n-1)\alpha} \right)^2 - \left(\frac{\sin n\alpha}{n\alpha} \right)^2 \right\}$$

therefore approaches a limiting value, as α decreases, that cannot be larger than

$$\delta \left\{ 1 + \frac{1}{\pi} + \frac{1}{\pi^2} \right\},$$

hence must be zero. Consequently

$$\frac{F(x+2\alpha) - 2F(x) + F(x-2\alpha)}{4\alpha^2},$$

which is equal to

$$f(x) + \sum \epsilon_n \left\{ \left(\frac{\sin(n-1)\alpha}{(n-1)\alpha} \right)^2 - \left(\frac{\sin n\alpha}{n\alpha} \right)^2 \right\},$$

converges to $f(x)$ as α tends to 0. This proves our theorem for the case $\beta = \alpha$.

In order to prove the general case, let

$$F(x+\alpha+\beta)-2F(x)+F(x-\alpha-\beta)=(\alpha+\beta)^2(f(x)+\delta_1)$$
$$F(x+\alpha-\beta)-2F(x)+F(x-\alpha+\beta)=(\alpha-\beta)^2(f(x)+\delta_2),$$

from which

$$F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)$$
$$=4\alpha\beta f(x)+(\alpha+\beta)^2\delta_1-(\alpha-\beta)^2\delta_2.$$

As a consequence of the above result, δ_1 and δ_2 become infinitely small when α and β do. Then

$$\frac{(\alpha+\beta)^2}{4\alpha\beta}\delta_1-\frac{(\alpha-\beta)^2}{4\alpha\beta}\delta_2$$

will also be infinitely small if the coefficients of δ_1 and δ_2 do not become infinitely large, which does not occur since β/α remains finite. Consequently,

$$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta}$$

converges to $f(x)$, as we wished to prove.

Theorem 2

$$\frac{F(x+2\alpha)+F(x-2\alpha)-2F(x)}{2\alpha}$$

tends to 0 with α for all x.

In order to prove this, we split the series

$$\sum A_n\left(\frac{\sin n\alpha}{n\alpha}\right)^2$$

into three parts. The first contains all terms up to a fixed index m, from which term on the A_n are always smaller than ϵ. The second contains all of the following terms for which $n\alpha \leq$ a fixed number c. Then the third includes the rest of the series. It is then easy to see that if α decreases infinitely, the sum of the first finite part remains finite, that is, $<$ a fixed number Q; the second $< \epsilon\,\frac{c}{\alpha}$; and the third

$$<\epsilon\sum_{c<n\alpha}\frac{1}{n^2\alpha^2}<\frac{\epsilon}{\alpha c}.$$

Consequently

$$\frac{F(x+2\alpha)+F(x-2\alpha)-2F(x)}{2\alpha} = 2\alpha \sum A_n \left(\frac{\sin n\alpha}{n\alpha}\right)^2$$
$$< 2\left(Q\alpha + \epsilon\left(c+\frac{1}{c}\right)\right)$$

from which the theorem follows.

Theorem 3 *Let b and c denote two arbitrary constants with $c > b$. Let $\lambda(x)$ denote a function which is always continuous together with its first derivative between b and c, is 0 at the boundaries, and for which the second derivative does not have infinitely many maxima and minima. Then the integral*

$$\mu^2 \int_b^c F(x)\cos\mu(x-a)\,\lambda(x)\,dx,$$

is eventually less than any given number, if μ grows to infinity.

If one replaces $F(x)$ by its series expression, then one obtains for

$$\mu^2 \int_b^c F(x)\cos\mu(x-a)\,\lambda(x)\,dx$$

the series (Φ)

$$\mu^2 \int_b^c \left(C + C'x + A_0\frac{x^2}{2}\right)\cos\mu(x-a)\,\lambda(x)\,dx$$
$$-\sum_{n=1}^{\infty}\frac{\mu^2}{n^2}\int_b^c A_n\cos\mu(x-a)\,\lambda(x)\,dx.$$

Now $A_n\cos\mu(x-a)$ can clearly be expressed as an aggregate of

$$\cos(\mu+n)(x-a),\ \sin(\mu+n)(x-a),\ \cos(\mu-n)(x-a),\ \sin(\mu-n)(x-a).$$

Denote the sum of the first two terms by $B_{\mu+n}$, and the sum of the last two terms by $B_{\mu-n}$. Then $A_n\cos\mu(x-a) = B_{\mu+n} + B_{\mu-n}$,

$$\frac{d^2B_{\mu+n}}{dx^2} = -(\mu+n)^2B_{\mu+n}, \quad \frac{d^2B_{\mu-n}}{dx^2} = -(\mu-n)^2B_{\mu-n},$$

and, with increasing n, $B_{\mu+n}$ and $B_{\mu-n}$ become infinitely small, whatever x is.

Thus the general term of the series (Φ),

$$-\frac{\mu^2}{n^2}\int_b^c A_n \cos\mu(x-a)\,\lambda(x)\,dx,$$

is equal to

$$\frac{\mu^2}{n^2(\mu+n)^2}\int_b^c \frac{d^2 B_{\mu+n}}{dx^2}\,\lambda(x)\,dx + \frac{\mu^2}{n^2(\mu-n)^2}\int_b^c \frac{d^2 B_{\mu-n}}{dx^2}\,\lambda(x)\,dx.$$

After two integrations by parts, in which one first considers $\lambda(x)$ and then $\lambda'(x)$ as constant, we obtain

$$\frac{\mu^2}{n^2(\mu+n)^2}\int_b^c B_{\mu+n}\,\lambda''(x)\,dx + \frac{\mu^2}{n^2(\mu-n)^2}\int_b^c B_{\mu-n}\,\lambda''(x)\,dx,$$

since $\lambda(x)$ and $\lambda'(x)$, and hence also the terms standing outside the integral sign, will be 0 at the limits.

It is now easy to convince ourselves that $\int_b^c B_{\mu\pm n}\,\lambda''(x)\,dx$ becomes infinitely small when μ grows to infinity, whatever n may be. For this expression is equal to an aggregate of the integrals

$$\int_b^c \cos(\mu\pm n)(x-a)\,\lambda''(x)\,dx, \qquad \int_b^c \sin(\mu\pm n)(x-a)\,\lambda''(x)\,dx,$$

and if $\mu\pm n$ becomes infinitely large, these integrals tend to 0. However, if $\mu\pm n$ does not become infinitely large because n is infinitely large, their coefficients in these expressions are infinitely small.

Clearly, to prove our theorem it therefore suffices that the sum

$$\sum \frac{\mu^2}{(\mu-n)^2 n^2}$$

extended over all values of n which satisfy $n < -c'$, $c'' < n < \mu - c'''$, $\mu + c^{IV} < n$, remains finite when μ becomes infinitely large for any choice of quantities c. For, except for the terms for which $-c' < n < c''$, $\mu - c''' < n < \mu + c^{IV}$, which clearly become infinitely small and are of finite number, the series (Φ) clearly remains smaller than the sum multiplied by the largest value of $\int_b^c B_{\mu\pm n}\,\lambda''(x)\,dx$, which becomes infinitely small.

However, if $c > 1$, the sum

$$\sum \frac{\mu^2}{(\mu - n)^2 n^2} = \frac{1}{\mu} \sum \frac{\frac{1}{\mu}}{\left(1 - \frac{n}{\mu}\right)^2 \left(\frac{n}{\mu}\right)^2},$$

within the limits above, is smaller than

$$\frac{1}{\mu} \int \frac{dx}{(1-x)^2 x^2},$$

taken from

$$-\infty \text{ to } -\frac{c' - 1}{\mu}, \quad \frac{c'' - 1}{\mu} \text{ to } 1 - \frac{c''' - 1}{\mu}, \quad 1 + \frac{c^{IV} - 1}{\mu} \text{ to } \infty.$$

For if we decompose the whole interval from $-\infty$ to ∞, starting from 0, into intervals of length $1/\mu$, and replace the function under the integral sign by the smallest value in each interval, we obtain all the terms of the series, since this function does not have maxima anywhere between the integration limits.

If the integration is carried out, we obtain,

$$\frac{1}{\mu} \int \frac{dx}{x^2 (1-x)^2} = \frac{1}{\mu} \left(-\frac{1}{x} + \frac{1}{1-x} + 2 \log x - 2 \log(1 - x) \right) + \text{const.},$$

and consequently between the above limits a number that does not become infinitely large with μ.

9.

We use these theorems to determine the following about the representation of a function by a trigonometric series whose terms tends to 0 for each value of the argument.

I. For a periodic function of period 2π to be represented by a trigonometric series whose terms eventually become infinitely small for each value of x, there must exist a continuous function $F(x)$ for which

$$\frac{F(x + \alpha + \beta) - F(x + \alpha - \beta) - F(x - \alpha + \beta) + F(x - \alpha - \beta)}{4\alpha\beta},$$

converges to $f(x)$ as α and β become infinitely small with their ratio remaining finite.

Furthermore, with increasing μ,

$$\mu^2 \int_b^c F(x) \cos \mu(x-a)\, \lambda(x)\, dx$$

must eventually become infinitely small as μ increases, if $\lambda(x)$ and $\lambda'(x)$ are 0 at the integration limits and always continuous between them, and $\lambda''(x)$ does not have infinitely many maxima and minima.

II. Conversely, if both these conditions are satisfied, then there is a trigonometric series in which the coefficients eventually become infinitely small and which represents the function, wherever it converges.

For the proof, determine the numbers C' and A_0 so that

$$F(x) - C'x - A_0 \frac{x^2}{2}$$

is a periodic function of period 2π, and expand this by Fourier's method into a trigonometric series

$$C - \frac{A_1}{1} - \frac{A_2}{4} - \frac{A_3}{9} - \cdots .$$

Here we let

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(F(t) - C't - A_0 \frac{t^2}{2} \right) dt = C,$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \left(F(t) - C't - A_0 \frac{t^2}{2} \right) \cos n(x-t)\, dt = -\frac{A_n}{n^2}.$$

Then, by agreement,

$$A_n = -\frac{n^2}{\pi} \int_{-\pi}^{\pi} \left(F(t) - C't - A_0 \frac{t^2}{2} \right) \cos n(x-t)\, dt$$

must eventually become infinitely small with increasing n. It follows by Theorem 1 of the preceding section that the series

$$A_0 + A_1 + A_2 + \cdots$$

converges to the function $f(x)$, wherever it converges.

III. Let $b < x < c$, and $\varrho(t)$ be a function such that $\varrho(t)$ and $\varrho'(t)$ are 0 for $t = b$ and $t = c$ and are continuous between those values, and such that

$\varrho''(t)$ does not have infinitely many maxima and minima, and, furthermore, such that for $t = x$, $\varrho(t) = 1$, $\varrho'(t) = 0$, $\varrho''(t) = 0$, $\varrho'''(t)$ and $\varrho^{IV}(t)$ are finite and continuous. Then the difference between the series

$$A_0 + A_1 + \cdots + A_n$$

and the integral

$$\frac{1}{2\pi}\int_b^c F(t)\,\frac{d^2\frac{\sin\frac{2n+1}{2}(x-t)}{\sin\frac{(x-t)}{2}}}{dt^2}\,\varrho(t)\,dt$$

tends to zero with increasing n. Hence the series

$$A_0 + A_1 + A_2 + \cdots$$

will converge or not, depending on whether

$$\frac{1}{2\pi}\int_b^c F(t)\frac{d^2\frac{\sin\frac{2n+1}{2}(x-t)}{\sin\frac{(x-t)}{2}}}{dt^2}$$

approaches a limit with increasing n, or not.

In fact,

$$A_1 + A_2 + \ldots + A_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(F(t) - C't - A_0\frac{t^2}{2}\right)\sum_{k=1}^{n} -k^2\cos k(x-t)\,dt.$$

Since

$$2\sum_{k=1}^{n} -k^2\cos k(x-t) = 2\sum_{k=1}^{n}\frac{d^2\cos k(x-t)}{dt^2} = \frac{d^2\frac{\sin\frac{2n+1}{2}(x-t)}{\sin\frac{(x-t)}{2}}\frac{(x-t)}{2}}{dt^2},$$

$$A_1 + A_2 + \cdots + A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(F(t) - C't - A_0\frac{t^2}{2}\right)\frac{d^2\frac{\sin\frac{2n+1}{2}(x-t)}{\sin\frac{(x-t)}{2}}}{dt^2}\,dt.$$

Now, by Theorem 3 of the preceding section,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(F(t) - C't - A_0\frac{t^2}{2}\right)\frac{d^2\frac{\sin\frac{2n+1}{2}(x-t)}{\sin\frac{(x-t)}{2}}}{dt^2}\lambda(t)\,dt$$

tends to 0 with infinitely increasing n if $\lambda(t)$ along with its first derivative is continuous, $\lambda''(t)$ does not have infinitely many maxima and minima, and for $t = x$, $\lambda(t) = 0$, $\lambda'(t) = 0$, $\lambda''(t) = 0$, $\lambda'''(t)$ and $\lambda^{IV}(t)$ are finite and continuous.

Set $\lambda(t)$ equal to 1 outside the boundaries b, c and $1 - \varrho(t)$ within those boundaries, which is clearly allowable. It follows that the difference between the series $A_1 + \cdots + A_n$ and the integral

$$\frac{1}{2\pi}\int_b^c \left(F(t) - C't - A_0\frac{t^2}{2}\right)\frac{d^2\frac{\sin\frac{2n+1}{2}(x-t)}{\sin\frac{(x-t)}{2}}}{dt^2}\,\varrho(t)\,dt$$

tends to 0 with increasing n. We easily see, by integration by parts, that

$$\frac{1}{2\pi}\int_b^c \left(C't + A_0\frac{t^2}{2}\right)\frac{d^2\frac{\sin\frac{2n+1}{2}(x-t)}{\sin\frac{(x-t)}{2}}}{dt^2}\,\varrho(t)\,dt,$$

converges to A_0 when n becomes infinitely large, and we obtain the above theorem.

10.

It has emerged from the investigation that if the coefficients of the series Ω tend to 0, then the convergence of the series for a particular value of x depends only on the behavior of the function $f(x)$ in the immediate neighborhood of this value.

Whether the coefficients of the series eventually become infinitely small, will in many cases not be decided by their expression as a definite integral, but in other ways. One case should be emphasized where the determination can be made immediately from the nature of the function. Namely, suppose the function $f(x)$ is everywhere finite and integrable.

In this case, we split the whole interval $-\pi$ to π into a sequence of pieces of length

$$\delta_1,\ \delta_2,\ \delta_3, \ldots$$

and denote by D_1 the greatest fluctuation of the function in the first, by D_2 the greatest fluctuation in the second, and so on. Then

$$\delta_1 D_1 + \delta_2 D_2 + \delta_3 D_3 + \cdots$$

must become infinitely small when the δ's become infinitely small together.

Consider the integral $\int_{-\pi}^{\pi} f(x)\sin n(x-a)\,dx$, which, apart from the factor $1/\pi$, gives the coefficients of the series, or what is the same thing, $\int_a^{a+2\pi} f(x)\sin n(x-a)\,dx$. We split this integral beginning at $x = a$, into integrals of range $2\pi/n$. Then each integral contributes to the sum a quantity less than $2/n$ multiplied by the greatest fluctuation in its interval, and their sum is hence smaller than a number, which by assumption must become infinitely small with $2\pi/n$.

In fact, these integrals have the form

$$\int_{a+\frac{s}{n}2\pi}^{a+\frac{s+1}{n}2\pi} f(x)\sin n(x-a)\,dx.$$

The sine is positive in the first half, and negative in the second. Denoting the largest value of $f(x)$ in the interval of integration by M and the smallest by m, it is obvious that the integral is bigger if we replace $f(x)$ by M in the first half and by m in the second. The integral is smaller if $f(x)$ is replaced by m in the first half and M in the second. In the first case we obtain the value $\frac{2}{n}(M-m)$; in the other $\frac{2}{n}(m-M)$. Hence the absolute value of the integral is smaller than $\frac{2}{n}(M-m)$, and the integral

$$\int_a^{a+2\pi} f(x)\sin n(x-a)\,dx$$

is smaller than

$$\frac{2}{n}(M_1-m_1)+\frac{2}{n}(M_2-m_2)+\frac{2}{n}(M_3-m_3)+\cdots,$$

where M_s denotes the largest value of $f(x)$ in the s-th interval and m_s the smallest. However, if $f(x)$ is integrable, this sum must become infinitely small as n goes to infinity, and the lengths of the intervals $2\pi/n$ become infinitely small.

In the case under discussion, then, the coefficients of the series become infinitely small.

11.

We must still examine the case where the terms of the series Ω eventually become infinitely small for an argument value x, without this occurring for each value of the argument. This case can be reduced to the previous one.

Namely, adding the terms of equal rank in the series for values $x + t$ and $x - t$, we obtain the series

$$2A_0 + 2A_1 \cos t + 2A_2 \cos 2t + \cdots .$$

In this series the terms for each value of t eventually become infinitely small, and the previous analysis can be applied.

For this purpose, denote the value of the infinite series

$$C + C'x + A_0 \frac{x^2}{2} + A_0 \frac{t^2}{2} - A_1 \frac{\cos t}{1} - A_2 \frac{\cos 2t}{4} - A_3 \frac{\cos 3t}{9} - \cdots$$

by $G(t)$, so that wherever the series $F(x+t)$ and $F(x-t)$ converge,

$$\frac{F(x+t) + F(x-t)}{2} = G(t).$$

We have the following:

I. If the terms of the series Ω tend to 0 for an argument value x, then

$$\mu^2 \int_b^c G(t) \cos \mu(t-a)\, \lambda(t)\, dt,$$

must eventually become infinitely small with increasing μ, where λ is a function as designated in §9. The value of the integral consists of the components

$$\mu^2 \int_b^c \frac{F(x+t)}{2} \cos \mu(t-a)\, \lambda(t)\, dt \quad \text{and} \quad \mu^2 \int_b^c \frac{F(x-t)}{2} \cos \mu(t-a)\, \lambda(t)\, dt,$$

provided that these expressions have a value. Hence the integral tends to 0 because of the behavior of the function F at two places lying symmetrically on both sides of x. It should be noted, however, that the positions must be situated where each component is not itself infinitely small. For then the terms of the series would eventually become infinitely small for each value of the argument. Thus the contribution of the positions situated symmetrically on both sides of x must cancel in such a way that their sum becomes infinitely small for an infinite μ. It follows from this that the series Ω can converge only for those values of x at the midpoint of places where

$$\mu^2 \int_b^c F(x) \cos \mu(x-a)\, \lambda(x)\, dx$$

does not become infinitely small for an infinite μ. Clearly the number of those places must be infinitely large if the trigonometric series whose coefficients are not infinitely decreasing is to converge for an infinite number of argument values.

Conversely,

$$A_n = -n^2 \frac{2}{\pi} \int_0^\pi \left(G(t) - A_0 \frac{t^2}{2} \right) \cos nt \, dt$$

and thus A_n tends to 0 with increasing n, if

$$\mu^2 \int_b^c G(t) \cos \mu(t-a) \, \lambda(t) \, dt$$

always becomes infinitely small for infinite μ.

II. If the terms of the series Ω eventually become infinitely small for an argument value x, then whether or not the series converges depends only on the behavior of the function $G(t)$ for infinitely small t. Indeed, the difference between

$$A_0 + A_1 + \cdots + A_n$$

and the integral

$$\frac{1}{\pi} \int_0^b G(t) \frac{d^2 \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}}}{dt^2} \varrho(t) \, dt$$

tends to 0 with increasing n, where b is a constant, however small, between 0 and π, and $\varrho(t)$ denotes a function such that $\varrho(t)$ and $\varrho'(t)$ are everywhere continuous and zero for $t = b$, $\varrho''(t)$ does not have infinitely many maxima and minima and for $t = 0$, $\varrho(t) = 1$, $\varrho'(t) = 0$, $\varrho''(t) = 0$, and $\varrho'''(t)$, $\varrho^{IV}(t)$ are finite and continuous.

12.

The conditions for the representation of a function by a trigonometric series can certainly be restricted a little further. Hence our examination can be extended somewhat further without special hypothesis on the nature of the functions. For example, in the last theorem the condition $\varrho''(t) = 0$ can be omitted if in the integral

$$\frac{1}{\pi} \int_0^b G(t) \frac{d^2 \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}}}{dt^2} \varrho(t) \, dt,$$

$G(t)$ is replaced by $G(t) - G(0)$. However, nothing essential is gained.

Therefore we turn to the consideration of particular cases. We will first examine a function which does not have infinitely many maxima and minima. We seek to give a complete solution for this case, which is possible by the work of Dirichlet.

It is noted above that such a function is everywhere integrable where it is not infinite, and clearly that can only occur for a finite number of argument values. Also by Dirichlet's proof, in the integral expressions for the nth term of the series and for the sum of the first n terms, the contribution from all intervals eventually become infinitely small with increasing n, with the exception of those where the function becomes infinite and the infinitesimal interval enclosing the argument of the series. Further, by Dirichlet's proof,

$$\int_x^{x+b} f(t) \frac{\sin \frac{2n+1}{2}(x-t)}{\sin \frac{x-t}{2}} dt$$

will converge to $\pi f(x+0)$ as n tends to infinity, if $0 < b < \pi$ and $f(t)$ is not infinite between the integration limits. Indeed nothing more is needed when one omits the unnecessary hypothesis that the function is continuous. Hence it only remains to examine, for this integral, in which cases the contribution of the places where the function becomes infinite tends to 0 with increasing n. This investigation is still incomplete. But Dirichlet showed in passing that this takes place if the function to be represented is integrable. This hypothesis is unnecessary.

We have seen above that if the terms of the series Ω tend to zero for each value of x, the function $F(x)$ whose second derivative is $f(x)$ must be finite and continuous and that

$$\frac{F(x+\alpha) - 2F(x) + F(x-\alpha)}{\alpha}$$

always becomes infinitely small with α. Now, if $F'(x+t) - F'(x-t)$ does not have infinitely many maxima and minima, then as t tends to zero it must converge to a limit L, or become infinitely large. It is clear that likewise,

$$\frac{1}{\alpha}\int_0^{\alpha} (F'(x+t) - F'(x-t))\, dt = \frac{F(x+\alpha) - 2F(x) + F(x-\alpha)}{\alpha}$$

must converge to L or to infinity and hence can only become infinitely small if $F'(x+t) - F'(x-t)$ converges to zero. Therefore $f(a+t) + f(a-t)$ must always be integrable up to $t = 0$ if $f(x)$ is infinitely large for $x = a$. This suffices for

$$\left(\int_b^{a-\epsilon} + \int_{a+\epsilon}^{c}\right) dx(f(x)\cos n(x-a))$$

to converge with decreasing ϵ, and to tend to 0 with increasing n. Furthermore, since $F(x)$ is finite and continuous, then $F'(x)$ can be integrated up to $x = a$ and $(x-a)F'(x)$ becomes infinitely small with $x - a$, if this function does not have infinitely many maxima and minima. It follows that

$$\frac{d(x-a)F'(x)}{dx} = (x-a)f(x) + F'(x),$$

and hence $(x-a)f(x)$, can be integrated up to $x = a$. Therefore $\int f(x)\sin n(x-a)\,dx$ can be integrated up to $x = a$. For the coefficients of the series eventually to become infinitely small, clearly it is only necessary that

$$\int_b^c f(x)\sin n(x-a)\,dx, \quad \text{where} \quad b < a < c,$$

tends to 0 with increasing n. If one sets

$$f(x)(x-a) = \phi(x),$$

then for an infinite n, if this function does not have infinitely many maxima and minima,

$$\int_b^c f(x)\sin n(x-a)\,dx = \int_b^c \frac{\phi(x)}{x-a}\sin n(x-a)\,dx = \pi\frac{\phi(a+0)+\phi(a-0)}{2},$$

as Dirichlet has shown. Therefore

$$\phi(a+t) + \phi(a-t) = f(a+t)t - f(a-t)t$$

must tend to 0 with t. Since

$$f(a+t) + f(a-t)$$

is integrable up to $t = 0$ and consequently

$$f(a+t)t + f(a-t)t$$

also becomes infinitely small with decreasing t, then $f(a+t)t$ as well as $f(a-t)t$ tend to 0 with decreasing t. Apart from the functions which have infinitely many maxima and minima, it is necessary and sufficient for the representation of a function $f(x)$ by a trigonometric series whose coefficients tend to 0, that if f become infinite for $x = a$, then $f(a+t)t$ and $f(a-t)t$ tend to 0 with t and $f(a+t) + f(a-t)$ is integrable up to $t = 0$.

A function $f(t)$ which does not have infinitely many maxima and minima can be represented only for finitely many values of the argument by a trigonometric series whose coefficients do not eventually tend to 0. For

$$\mu^2 \int_b^c F(x) \cos \mu(x-a)\, \lambda(x)\, dx$$

fails to tend to 0 as μ becomes infinite, at only a finite number of values. Hence it is unnecessary to consider this further.

13.

Concerning functions with infinitely many maxima and minima, it is probably not superfluous to note that there exists such a function $f(x)$, everywhere integrable, that cannot be represented by a Fourier series. This occurs, for example, if

$$f(x) = \frac{d\left(x^\nu \cos \frac{1}{x}\right)}{dx}, \quad \text{for } 0 \le x \le 2\pi, \text{ and } 0 < \nu < 1/2.$$

For the contribution in the integral $\int_0^{2\pi} f(x) \cos n(x-a)\, dx$ with increasing n of those places where x is close to $\sqrt{\frac{1}{n}}$ is, generally speaking, eventually infinitely large, so that the ratio of this integral to

$$\frac{1}{2} \sin\left(2\sqrt{n} - na + \frac{\pi}{4}\right) \sqrt{\pi} n^{\frac{1-2\nu}{4}}$$

converges to 1, as we find by the method just described. In order to generalize the the example, and bring out the essence of the matter, let

$$\int f(x)\, dx = \phi(x) \cos \psi(x)$$

and assume that $\phi(x)$ is infinitely small for an infinitely small x, and $\psi(x)$ becomes infinitely large, and elsewhere these functions together with their derivatives are continuous and do not have infinitely many maxima and minima. Then

$$f(x) = \phi'(x) \cos \psi(x) - \phi(x)\psi'(x) \sin \psi(x),$$

and

$$\int f(x) \cos n(x-a)\, dx$$

is the sum of the four integrals

$$\frac{1}{2} \int \phi'(x) \cos(\psi(x) \pm n(x-a))\, dx,$$
$$-\frac{1}{2} \int \phi(x)\psi'(x) \sin(\psi(x) \pm n(x-a))\, dx.$$

Taking $\psi(x)$ positive, we consider the term

$$-\frac{1}{2} \int \phi(x)\psi'(x) \sin(\psi(x) + n(x-a))\, dx$$

and examine in this integral the place where the changes of sign of the sine follow one another most slowly. Let

$$\psi(x) + n(x-a) = y,$$

then this occurs where $\frac{dy}{dx} = 0$. Thus $x = \alpha$ with

$$\psi'(\alpha) + n = 0.$$

We therefore examine the behavior of the integral

$$-\frac{1}{2} \int_{\alpha-\epsilon}^{\alpha+\epsilon} \phi(x)\psi'(x) \sin y\, dx$$

in the case that ϵ becomes infinitely small for an infinite n, and introduce y as a variable. Let

$$\psi(\alpha) + n(\alpha - a) = \beta,$$

then for sufficiently small ϵ

$$y = \beta + \psi''(\alpha) \frac{(x-\alpha)^2}{2} + \cdots$$

and, indeed, $\psi''(\alpha)$ is positive, since $\psi(x)$ tends to $+\infty$ as x tends to 0. Furthermore,

$$\frac{dy}{dx} = \psi''(\alpha)(x-\alpha) = \pm\sqrt{2\psi''(\alpha)(y-\beta)},$$

depending on whether $x-\alpha > 0$ or < 0; and

$$\begin{aligned} -\frac{1}{2}\int_{\alpha-\epsilon}^{\alpha+\epsilon} & \phi(x)\psi'(x)\sin y\,dx \\ &= \frac{1}{2}\left(\int_{\beta+\psi''(\alpha)\frac{\epsilon^2}{2}}^{\beta} - \int_{\beta}^{\beta+\psi''(\alpha)\frac{\epsilon^2}{2}}\right)\frac{\phi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}\left(\sin y\frac{dy}{\sqrt{y-\beta}}\right) \\ &= -\int_0^{\psi''(\alpha)\frac{\epsilon^2}{2}}\sin(y+\beta)\,\frac{\phi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}\,\frac{dy}{\sqrt{y}}. \end{aligned}$$

Let ϵ decrease with increasing n so that $\psi''(\alpha)\epsilon^2$ becomes infinitely large. If

$$\int_0^{\infty}\sin(y+\beta)\,\frac{dy}{\sqrt{y}},$$

which is known to be $\sin(\beta+\pi/4)\sqrt{\pi}$, is not zero, then disregarding quantities of lower order,

$$-\frac{1}{2}\int_{\alpha-\epsilon}^{\alpha+\epsilon}\phi(x)\psi'(x)\sin(\psi(x)+n(x-a))\,dx = -\sin\left(\beta+\frac{\pi}{4}\right)\frac{\sqrt{\pi}\phi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}.$$

Hence, if the last quantity does not become infinitely small, its ratio to

$$\int_0^{2\pi} f(x)\cos n(x-a)\,dx$$

converges to 1 with an infinite increase of n, since the remaining contributions become infinitely small.

Assume that $\phi(x)$ and $\psi'(x)$ are of the same order as powers of x for infinitely small x, with $\phi(x)$ of the order of x^{ν} and $\psi'(x)$ of the order of $x^{-\mu-1}$, where we must have $\nu > 0$ and $\mu \geq 0$. Then for infinite n,

$$\frac{\phi(\alpha)\psi'(\alpha)}{\sqrt{2\psi''(\alpha)}}$$

has the same order as $\alpha^{\nu-\frac{\mu}{2}}$ and hence is not infinitely small when $\mu \geq 2\nu$. In general however, if $x\psi'(x)$ or, what is the same thing, if $\frac{\psi(x)}{\log x}$ is infinitely large for an infinitely small x, $\phi(x)$ can be taken so that $\phi(x)$ tends to 0 with x, while

$$\phi(x)\,\frac{\psi'(x)}{\sqrt{2\psi''(x)}} = \frac{\phi(x)}{\sqrt{-2\frac{d}{dx}\frac{1}{\psi'(x)}}} = \frac{\phi(x)}{\sqrt{-2\lim\frac{1}{x\psi'(x)}}}$$

will be infinitely large. Consequently $\int_x f(x)\,dx$ can be extended to $x = 0$, while

$$\int_0^{2\pi} f(x)\cos n(x-a)\,dx$$

does not become infinitely small for an infinite n. We see that the increases in the integral $\int_x f(x)\,dx$ as x tends to 0 cancel out because of the rapid changes of sign of the function $f(x)$, although their variation increases very rapidly in ratio to the change of x. However, the introduction here of the factor $\cos n(x-a)$ results in this increase being summable.

Just as, in the above, the Fourier series does not converge for a function in spite of the overall integrability, and the terms themselves eventually become infinitely large, it can happen that, despite the overall non-integrability of $f(x)$, between each two values of x, no matter how close, there are infinitely many values for which the series Ω converges.

An example is given by the function defined by the series

$$\sum_{n=1}^{\infty}\frac{(nx)}{n}$$

which exists for each rational value of x, where the meaning of (nx) is taken as in §6. This can be represented by the trigonometric series

$$\sum_{n=1}^{\infty}\frac{\Sigma^{\theta}-(-1)^{\theta}}{n\pi}\sin 2nx\pi,$$

where θ runs over the divisors of n. The function is not bounded in any interval, no matter how small, and hence is nowhere integrable.

Another example is obtained if in the series

$$\sum_{n=0}^{\infty}c_n\cos n^2x, \qquad \sum_{n=1}^{\infty}c_n\sin n^2x$$

$c_0, c_1, c_2, \ldots$ are positive numbers which always decrease and tend to 0, while $\sum_{s=1}^{n} c_s$ becomes infinitely large with n. For if the ratio of x to 2π is rational and in lowest terms has denominator m, then clearly the series converges or tends to infinity depending on whether

$$\sum_{n=0}^{m-1} \cos n^2 x, \qquad \sum_{n=0}^{m-1} \sin n^2 x$$

are zero or not. Both cases arise, by a well known theorem[33] on partitioning the circle, for infinitely many values of x between any two bounds, no matter how close.

The series Ω can converge in a range just as large, without the value of the series

$$C' + A_0 x - \sum \frac{1}{n^2} \frac{dA_n}{dx},$$

which one obtains by termwise integration of Ω, being integrable on any interval, however small.

For example, we expand the expression

$$\sum_{n=1}^{\infty} \frac{1}{n^3}(1-q^n) \log \left(\frac{-\log(1-q^n)}{q^n} \right),$$

where the logarithms are taken so that they vanish for $q = 0$, by increasing powers of q, and replace q by e^{xi}. The imaginary part is a trigonometric series whose second derivative with respect to x converges infinitely often on any interval, while its first derivative becomes infinite infinitely often.

In the same range, that is, between any two argument values no matter how close, a trigonometric series can also converge infinitely often when its coefficients do not tend to 0. A simple example of such a series is given by $\sum_{n=1}^{\infty} \sin(n! x\pi)$, where as usual,

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

This not only converges for each rational value of x, for which it changes into a finite sum, but also for an infinite number of irrationals, of which the simplest are $\sin 1$, $\cos 1$, $2/e$ and their multiples, odd multiples of e, $\frac{e-\frac{1}{e}}{4}$, and so on.

[33]Disquis. ar. p. 636, §356. (Gauss, *Werke*, vol. I, p. 442.)

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