MA 527 second midterm review problems
Hopefully final version as of November 8th

- The second midterm will be on Monday, November 13th, from 8 to 9 pm, in MTHW 210.
- It will cover all the material we have done from sections 6.1–6.7, 11.1–11.5, and 12.3
- Most of the problems on the exam will be closely based on ones from the list below (but the actual exam will be much shorter).
- For each problem, you must explain your reasoning.
- The last page is a reference page, which will also be provided on the actual exam.
- Note that these are not arranged in order of difficulty!

1. Find $y(4)$, where $y(t)$ solves
$$y'' + 6y' + 9y = \delta(t-3), \qquad y(0) = y'(0) = 0.$$

2. Find all values of $a > 0$ such that solution to the initial value problem
$$y''(t) + ay(t) = \delta(t-1) + \delta(t-2), \qquad y(0) = y'(0) = 0,$$
obeys $y(t) \equiv 0$ for all $t > a$.

   Hint: It may be helpful to use the identity $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$.

3. (a) Find the Laplace transform of $f * g$, where $f(t) = e^t u(t-1)$ and $g(t) = t\sin t$.

   (b) Use the result of part (a) to evaluate
$$\int_0^\infty e^{-5t}\int_1^t e^\tau (t-\tau)\sin(t-\tau)d\tau dt.$$

4. Let f be the function of period 2 such that
$$f(x) = \begin{cases} 1+x, & -1 \le x < 0 \\ -x, & 0 \le x < 1. \end{cases}$$

   (a) Find the Fourier series for f.

   (b) Let $g(x)$ be the sum of the first three terms of this Fourier series. Evaluate the error $E = \int_{-1}^1 ((f(x) - g(x))^2 dx$.

5. Find a Fourier series for $f(x) = x^2$ valid when $0 < x < 1$ that contains

   (a) no sine terms,

   (b) no cosine terms,

   (c) both sine and cosine terms.

   Write out the first four nonzero terms in each case.

6. (a) For each $n > 0$, find a constant A_n such that $y(t) = A_n \sin \pi n t$ is a solution to
$$y''(t) + 81y(t) = \sin \pi n t.$$

   (b) Let $f(t)$ be the odd function of period 2 which is identically 1 when $0 < t < 1$. Find the Fourier series of $f(t)$ and write out the first three nonzero terms of the series.

   (c) Find the Fourier series of the periodic solution to
$$y''(t) + 81y(t) = f(t),$$
   and write out the first three nonzero terms of the series.

7. Consider the boundary value problem $x^2y'' + 5xy' - 8y + \lambda y = 0$, $y(1) = y(2) = 0$.

   (a) Find $a > 0$ such that multiplying the equation by x^a puts it into Sturm–Liouville form.

   (b) For what inner product on $[1, 2]$ are the eigenfunctions of this Sturm–Liouville problem orthogonal? (You do not need to find the eigenvalues or eigenfunctions.)

8. Find the eigenvalues and eigenfunctions of
$$y'' + 4y' + (\lambda + 8)y = 0, \qquad y(0) = y(2) = 0.$$
   With respect to which inner product on the interval $[0, 2]$ are the eigenfunctions orthogonal?

9. Find a constant a such that -1 is an eigenvalue of the Sturm–Liouville problem
$$y'' + \lambda y = 0, \qquad y'(0) = 0, \qquad y(\ln 2) + ay'(\ln 2) = 0.$$

What eigenfunctions correspond to this eigenvalue?

10. In class we derived the formulas

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos nt + B_n^* \sin nt) \sin nx$$

for the solution to

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t), \qquad u(0,t) = u(\pi,t) = 0,$$

and

$$u(x,t) = B_0 + B_0^* t + \sum_{n=1}^{\infty} (B_n \cos nt + B_n^* \sin nt) \cos nx$$

for the solution to

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t), \qquad \partial_x u(0,t) = \partial_x u(\pi,t) = 0.$$

(We also talked about how to write the coefficients B_n, B_n^* in terms of given initial conditions, but you don't need to worry about this for this problem.) Derive the corresponding formula for

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t), \qquad u(0,t) = \partial_x u(\pi,t) = 0.$$

11. When $0 \le x \le 9$, define $f(x)$ by

$$f(x) = \begin{cases} 0, & 0 \le x < 3 \\ 1, & 3 \le x \le 6 \\ 0, & 6 < x \le 9. \end{cases}$$

(a) Sketch the graph of $f(x)$ for $0 \le x \le 9$.

(b) Let $u(x,t) = \frac{1}{2}(f^*(x-t) + f^*(x+t))$, where f^* is the odd extension of f with period 18. Sketch the graphs of $u(x,n)$ for various choices of n, where n is a nonnegative integer.

(c) Repeat part (b) with $u(x,t) = \frac{1}{2}(f^{**}(x-t) + f^{**}(x+t))$, where f^{**} is the even extension of f with period 18.

Reference page

- The Laplace transform is defined by $\mathcal{L}[f(t)](s) = F(s) = \int_0^\infty e^{-ts} f(t)dt$.

$f(t)$	$F(s)$
t^n	$n!s^{-n-1}$
$e^{at}f(t)$	$F(s-a)$
y'	$sY(s) - y(0)$
y''	$s^2Y(s) - sy(0) - y'(0)$
$\cos bt$	$s/(s^2+b^2)$
$\sin bt$	$b/(s^2+b^2)$
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$\delta(t-a)$	e^{-as}
$f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$tf(t)$	$-F'(s)$

- If f has period $2L$, then the Fourier series for f is $a_0 + \sum_{n=1}^\infty a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)$, with $a_0 = (2L)^{-1} \int_{-L}^L f(x)dx$, $a_n = L^{-1} \int_{-L}^L f(x)\cos(n\pi x/L)dx$, $b_n = L^{-1} \int_{-L}^L f(x)\sin(n\pi x/L)dx$.
- If $g(x) = a_0 + \sum_{n=1}^N a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)$, then $E = \int_{-L}^L (f(x) - g(x))^2 dx = \int_{-L}^L f(x)^2 dx - 2La_0^2 - L\sum_1^N a_n^2 + b_n^2$.
- The Sturm–Liouville problem

$$(py')' + qy + \lambda ry = 0$$

with boundary conditions $k_1 y(a) + k_2 y'(a) = 0$ and $l_1 y(b) + l_2 y'(b) = 0$ has eigenfunctions orthogonal with respect to the inner product $(f, g) = \int_a^b f(x)g(x)r(x)dx$.
- The differential equation $y'' + ay' + by = 0$ has the general solution
  1. $y(x) = Ae^{r_1 x} + Be^{r_2 x}$ if r_1 and r_2 are the distinct roots of $r^2 + ar + b = 0$,
  2. $y(x) = Ae^{r_0 x} + Bxe^{r_0 x}$ if r_0 is the unique repeated root of $r^2 + ar + b = 0$,
  3. $y(x) = Ae^{cx}\cos(\omega x) + Be^{cx}\sin(\omega x)$ if $c \pm i\omega$ are the distinct roots of $r^2 + ar + b = 0$.
- $\sin 0 = 0$, $\sin \pi/2 = 1$, $\sin \pi = 0$, $\sin 3\pi/2 = -1$, $\sin 2\pi = 0$, etc.
- $\cos 0 = 1$, $\cos \pi/2 = 0$, $\cos \pi = -1$, $\cos 3\pi/2 = 0$, $\cos 2\pi = 1$, etc.