MA 224 - Exam 3 SOLUTIONS

1. $f(x, y) = xy + x + y$, $g(x, y) = x + y = 30$.

   Set up Lagrange equations:

   $f_x = \lambda g_x \rightarrow y + 1 = \lambda 1$

   $f_y = \lambda g_y \rightarrow x + 1 = \lambda 1$

   $x + y = 30$

   We set $y + 1 = \lambda = x + 1$ so $y + 1 = x + 1$, so $x = y$. We plug this into the constraint equation so $x + y = 30$ becomes $x + x = 30$ so $2x = 30$, so $x = 15$ and $y = 15$.

   Now we test $(15, 15)$ to ensure it is a max: $(30, 0)$ is a point that satisfies the contraint, and plugging it into $f(x, y)$ gives $f(30, 0) = 30 * 0 + 30 + 0 = 30$. This is smaller than $f(15, 15) = 15 * 15 + 15 + 15 = 255$, so we found a max at $(15, 15)$, and the max value is 255 thousand.

2. $f(x, y) = x^2 y$, $g(x, y) = x^2 + 2y^2 = 6$.

   Set up Lagrange equations:

   $f_x = \lambda g_x \rightarrow 2xy = \lambda 2x$

   $f_y = \lambda g_y \rightarrow x^2 = \lambda 4y$

   $x^2 + 2y^2 = 6$

   If we assume $x \neq 0$ then we can divide by x, so the equation $2xy = \lambda 2x$ becomes $y = \lambda$. If we plug this into our other equation, $x^2 = \lambda 4y$ then it becomes $x^2 = 4y^2$. We can substitute this into the constraint equation, $x^2 + 2y^2 = 6$ to get $6y^2 = 6$. Dividing by 6 gives $y^2 = 1$, and so $y = \pm 1$.

   If we plug these into the constraint equation we get $x^2 + 2 * 1 = 6$, so $x^2 = 4$. Then $x = \pm 2$. So we have the critical points $(2, 1), (-2, 1), (2, -1), (-2, -1)$.

   Now consider the case that $x = 0$. In order for the point (x, y) to satisfy the constraint, we need $x^2 + 2y^2 = 6$, so $0 + 2y^2 = 6$, so $y^2 = 3$, so $y = \pm\sqrt{3}$. But this doesn't satisfy the other Lagrange equations (when we plug it into $x^2 = \lambda 4y$), so it's not really a critical point.

   Finally, we need to test the critical points $(2, 1), (-2, 1), (2, -1), (-2, -1)$ to see which are max and which are mins, and what those max and min values are.

   Plugging the points into $f(x, y)$ gives

   Maxes: $f(2, 1) = 4$, $f(-2, 1) = 4$

   Mins: $f(2, -1) = -4$, $f(-2, -1) = -4$

3. (a)

   $$\int_0^1 \int_2^3 6ye^{xy} dx\ dy$$

   We do the inner integral first: $\int_2^3 6ye^{xy} dx = 6y \int_2^3 e^{xy} dx = 6y \left(\frac{1}{y} e^{xy} \right]_2^3 = 6 \left(e^{xy} \right]_2^3 = 6 \left(e^{3y} - e^{2y} \right)$.

   Now we plug this into the outer integral:

$\int_0^1 6\left(e^{3y}-e^{2y}\right)dy = 6\int_0^1 \left(e^{3y}-e^{2y}\right)dy = 6\left(\frac{1}{3}e^{3y}-\frac{1}{2}e^{2y}\right]_0^1 = 6\left[\left(\frac{1}{3}e^3-\frac{1}{2}e^2\right)-\left(\frac{1}{3}-\frac{1}{2}\right)\right].$
And distributing the 6 through gives

$(2e^3-3e^2)-(2-3)=2e^3-3e^2+1.$

(b)

$$\int_0^1\int_x^{x^2} 24xydy\, dx$$

Doing the inner integral first we get $\int_x^{x^2} 24xydy = 24x\int_x^{x^2} ydy = 24x\left(\frac{1}{2}y^2\right]_x^{x^2}$. Factoring out the $\frac{1}{2}$ gives $12x\left(y^2\right]_x^{x^2}$. Now plugging in x^2 and x into y^2 we get $12x\left((x^2)^2-(x)^2\right) = 12x\left(x^4-x^2\right) = 12x^5-12x^3$. Now we can plug this into the outer integral:

$\int_0^1 \left(12x^5-12x^3\right)dx = \left(2x^6-3x^4\right]_0^1$. Finally, evaluating at 1 and 0 gives

$[(2(1)-3(1))-(2(0)-3(0))] = -1$ as a final answer.

4. (a)

$$-1+\frac{1}{4}-\frac{2}{7}+\frac{6}{10}-\frac{24}{13}+\frac{120}{16}-\cdots$$

Since the terms alternate positive negative, we know $(-1)^n$ is a component. Since it starts as negative, when $n=0$ we need it to be negative, so the exponent should be $(-1)^{n+1}$ so that it's -1 when $n=0$.

The denominators are 1, 4, 7, 10, 13, 16 — the denominator goes up by 3 each time. To make the denominator go up by 3 each time n goes up by one, $3n$ must somehow be involved. But the denominator starts at 1 when $n=0$, so the denominator is $3n+1$.

Finally, the numerators are $1, 1, 2, 6, 24, 120$. Hopefully by now you can recognize this pattern as $0!, 1!, 2!, 3!, 4!, 5!$, so the numerator is $n!$, starting at $n=0$.

Putting it all together gives $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\cdot n!}{3n+1}$.

(b) $\frac{7^{n+1}}{6^{n-1}} = \frac{7^n}{6^n}\cdot\frac{7}{6^{-1}} = \left(\frac{7}{6}\right)^n 42$. So this is a geometric series with $r=\frac{7}{6}$, which is greater than 1, so by the geometric series convergence theorem this series diverges.

(c) $12e^{(.7)n} = 12\cdot\left(e^{.7}\right)^n$. So this is a geometric series with $r=e^{.7}$, which is greater than 1, so by the geometric series convergence theorem this series diverges.

(d) $e^{(-.1)n} = \left(e^{-.1}\right)^n$, so this is a geometric series with $r=e^{-.1}$ which is less than 1, and so by the geometric convergence theorem this series converges.

To figure out what it converges to, we use the other half of the geometric convergence theorem: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

The series we're working on starts at $n=2$, though, so we have some work to do. $\sum_{n=2}^{\infty} e^{(-.1)n} = \sum_{n=0}^{\infty} e^{(-.1)n} - e^{(-.1)\cdot 0} - e^{(-.1)\cdot 1} = \sum_{n=0}^{\infty} e^{(-.1)n} - e^0 - e^{(-.1)} = \frac{1}{1-e^{(-.1)}} - 1 - e^{(-.1)}$. Plugging into your calculator and rounding to 4 digits gives 8.6035.

5. (a) To write this function as a power series $\frac{2x}{3+x^2}$ we want to use the geometric series formula $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$. For this to work, we need $\frac{2x}{3+x^2}$ to look more like $\frac{1}{1-r}$. To make the numerator into a 1 we just factor out the $2x$ to get $2x\frac{1}{3+x^2}$. Then, factor a 3 out of the denominator: $2x\frac{1}{3+x^2} = 2x\frac{1}{3\left(1+\frac{x^2}{3}\right)} = \frac{2x}{3}\frac{1}{1+\frac{x^2}{3}}$. Finally, we need $1-r$ in the denominator, and to get this we change $1+\frac{x^2}{3}$ to $1-(-\frac{x^2}{3})$.

   Putting this all together we get $\frac{2x}{3+x^2} = \frac{2x}{3}\frac{1}{1-(-\frac{x^2}{3})}$. Now we can use the geometric series formula to write this as $\frac{2x}{3+x^2} = \frac{2x}{3}\frac{1}{1-(-\frac{x^2}{3})} = \frac{2x}{3}\left(\sum_{n=0}^{\infty}\left(-\frac{x^2}{3}\right)^n\right)$. Finally, bring the $\frac{2x}{3}$ inside: $\left(\sum_{n=0}^{\infty}\frac{2x}{3}\left(-\frac{x^2}{3}\right)^n\right)$. Then, distribute the exponent n: $\left(-\frac{x^2}{3}\right)^n = (-1)^n\frac{x^{2n}}{3^n}$. So now our series looks like $\sum_{n=0}^{\infty}\frac{2x}{3}\left(-\frac{x^2}{3}\right)^n = \sum_{n=0}^{\infty}\frac{2x}{3}(-1)^n\frac{x^{2n}}{3^n}$, and combining other terms we get $\sum_{n=0}^{\infty} 2(-1)^n\frac{x^{2n+1}}{3^{n+1}}$.

   (b) To find the radius and interval of convergence for the power series $\displaystyle\sum_{n=6}^{\infty}\frac{3^{n+1}(-2)^n x^n}{n!}$, we need to use the ratio test. First we compute this limit: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, call that limit L. The ratio test says the power series converges if $L<1$. To compute this limit first we'll simplify $\frac{a_{n+1}}{a_n} = \frac{3^{(n+1)+1}(-2)^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{3^{n+1}(-2)^n x^n} = \frac{3^{n+2}(-2)^{n+1}n!x^{n+1}}{3^{n+1}(-2)^n(n+1)!x^n} = \frac{3^{n+2}}{3^{n+1}}\cdot\frac{(-2)^{n+1}}{(-2)^n}\cdot\frac{n!}{(n+1)!}\cdot\frac{x^{n+1}}{x^n} = 3\cdot(-2)\frac{n!}{(n+1)!}\cdot x$. Then, since $(n+1)! = (n+1)\cdot n!$ by a property of factorials given in class, we have $\frac{n!}{(n+1)!} = n$. So the expression on the inside of the limit is $\left|\frac{3\cdot(-2)\cdot x}{n}\right| = \left|\frac{6x}{n}\right|$. Now we evaluate the limit: $\lim_{n\to\infty}\left|\frac{6x}{n}\right| = 0$ because the denominator becomes infinite. Now since this limit is 0 (which is always less than 1) the ratio test says the power series always converges. So the interval of convergence is $(-\infty,\infty)$, or all real numbers, and the radius of convergence is $R=\infty$.

6. Let $f(x) = (1+2x)e^x$. To give the Taylor series about $a=0$, we use the formula $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(x-0)^n = \frac{f^{(n)}(0)}{n!}x^n$. To find the values for $f^{(n)}(0)$, first we need to find the derivatives, and then plug in 0 for x.

   To take the derivatives, we have to use the product rule for derivatives. The derivative of $(1+2x)e^x$ is the derivative of $(1+2x)$ times e^x plus $(1+2x)$ times the derivative of e^x, which is $= 2\cdot e^x + (1+2x)e^x = 2e^x + e^x + 2xe^x = 3e^x + 2xe^x = (3+2x)e^x$. We get the rest of the derivatives the same way:

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$f^{(0)}(x) = (1+2x)e^x$	$f^{(0)}(0) = (1+0)e^0 = 1$
1	$f^{(1)}(x) = (3+2x)e^x$	$f^{(1)}(0) = (3+0)e^0 = 3$
2	$f^{(2)}(x) = (5+2x)e^x$	$f^{(2)}(0) = (5+0)e^0 = 5$
3	$f^{(3)}(x) = (7+2x)e^x$	$f^{(3)}(0) = (7+0)e^0 = 7$
$\vdots$	$\vdots$	$\vdots$
n	$f^{(n)}(x) = ((2n+1)+2x)e^x$	$f^{(n)}(0) = ((2n+1)+0)e^0 = 2n+1$

   So ultimately $f^{(n)}(0) = 2n+1$. The power series is $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{2n+1}{n!}x^n$