1. Use the formula for a geometric series to compute the following

$$\sum_{k=0}^{\infty}(-1)^k \frac{3^{k-2}}{6^{k+1}}$$

A. $\frac{1}{81}$

B. $-\frac{1}{27}$

C. $-\frac{1}{81}$

D. $\frac{1}{27}$

E. $\frac{2}{27}$

Find a:

First term, K=0: $(-1)^0 \frac{3^{0-2}}{6^{0+1}} = \frac{3^{-2}}{6} = \frac{1}{54}$

Find r: $\left(\frac{K=1}{K=0}\right)$: $\frac{\left((-1)^1 \frac{3^{1-2}}{6^{1+1}}\right)}{\left((-1)^0 \frac{3^{0-2}}{6^{0+1}}\right)} = (-1)\frac{3^{-1}}{6^2}\cdot\frac{6^1}{3^{-2}} = -\frac{1}{2}$

Geometric series $= \frac{a}{1-r} = \frac{1/54}{1--1/2} = \frac{1/54}{3/2} = \frac{2}{3\cdot 54} = \frac{1}{81}$

2. Determine the interval of absolute convergence for the power series

$$\sum_{k=1}^{\infty}\frac{3^{2k}x^k}{k^3}$$

A. $-1/6 < x < 1/6$

B. $-1/3 < x < 1/3$

C. $-9 < x < 9$

D. $-3 < x < 3$

E. $-1/9 < x < 1/9$

Ratio test: $\left|\frac{a_{k+1}}{a_k}\right|$ is

$\left|\frac{3^{2(k+1)} x^{k+1}}{(k+1)^3} \cdot \frac{k^3}{3^{2k} x^k}\right| = \left|\frac{3^{\cancel{2k}+2}\, k^3\, \cancel{x^{k+1}}}{\cancel{3^{2k}}\,(k+1)^3\, \cancel{x^k}}\right|$

simplify: $\left|\frac{3^2 x\, k^3}{(k+1)^3}\right|$. $\lim_{k\to\infty}\left|\frac{9x\, k^3}{(k+1)^3}\right| = |9x|$.

Set $|9x|<1 \Longrightarrow |x| < 1/9 \quad (-\frac{1}{9}, \frac{1}{9})$

3. A tennis ball is dropped from a height of 25 feet and bounces indefinitely, repeatedly rebounding to 60% of its previous height. How far does the tennis ball travel by the time it hits the ground the fourth time?

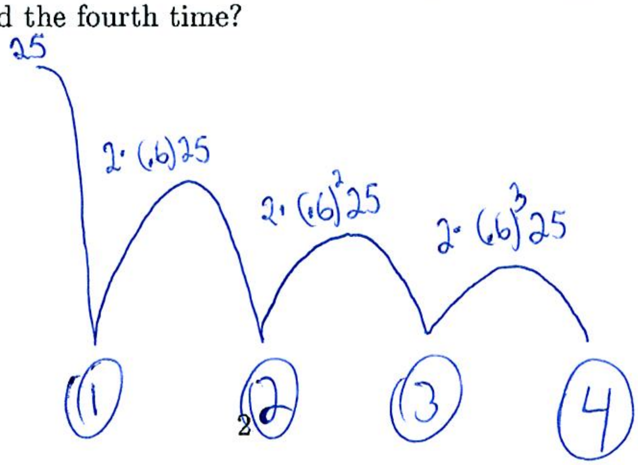
A. 125 feet

B. 108.8 feet

C. 100 feet

D. 83.8 feet

E. 54.4 feet



$25 + 2\cdot(.6)25 + 2(.6)^2 25 + 2\cdot(.6)^3 25 = 83.8$

4. Find the general solution of the following differential equation:

$$\frac{dy}{dx} = -5y^4$$

A. $y = -25x + C$

B. $y = -x^5 + C$

C. $y = \dfrac{1}{\sqrt[3]{15x + C}}$

D. $y = \sqrt[5]{25x} + C$

E. $y = \sqrt[5]{C - 25x}$

Separate: $dy = -5y^4\,dx \longrightarrow y^{-4}\,dy = -5dx$

integrate: $\int y^{-4}\,dy = \int -5dx$

$$\frac{y^{-3}}{-3} = -5x + C_1$$

$$\frac{1}{y^3} = 15x + C_2 \longrightarrow y^3 = \frac{1}{15x + C_2} \rightarrow y = \sqrt[3]{\frac{1}{15x+C}} = \frac{\sqrt[3]{1}}{\sqrt[3]{15x+C}}$$

5. Write the following infinite series in summation notation.

$$1 - \frac{4}{3} + \frac{7}{9} - \frac{10}{27} + \cdots$$

A. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3n-2}{3^n}$

B. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3n+1}{3^n}$

C. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{3n+1}{3^n}$

D. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3n-2}{3^{n-1}}$

E. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{3n-2}{3^{n-1}}$

numerator: $n=1$: 1, $n=2$: 4, $n=3$: 7, $n=4$: 10 ($+3$, $+3$, $+3$)

add 3 each time, so $1 + 3(n-1) =$ numerator $= 3n - 2$

denominator: $n=1$: 1, $n=2$: 3, $n=3$: 9, $n=4$: 27

powers of 3, so $3^{n-1} =$ denominator

because when $n=1$ the denominator is $1 = 3^0 = 3^{1-1}$

account for the sign changing: $1 - \frac{4}{3} + \frac{7}{9} - \frac{10}{27} + \cdots$

$+ \quad - \quad + \quad - \quad +$

$$\frac{3n-2}{3^{n-1}}(-1)^{n+1}$$

So we need $(-1)^{n+1}$, so that the first term is positive.

6. In a lab container there is a water based solution containing 75 grams of undissolved sugar. As the sugar is stirred and dissolved into the solution, the rate of change of the amount of sugar dissolved in the solution is proportional to the amount of undissolved sugar. $\frac{dS}{dt} = K \cdot (75-S)$

If $S(t)$ represents the amount of dissolved sugar at time t, where $S(t)$ is in grams, and k is the constant of proportionality, then the differential equation describing the given situation is:

A. $\frac{dS}{dt} = k\frac{S}{75}$

B. $\frac{dS}{dt} = k(S-75)$

C. $\frac{dS}{dt} = k(75-S)$

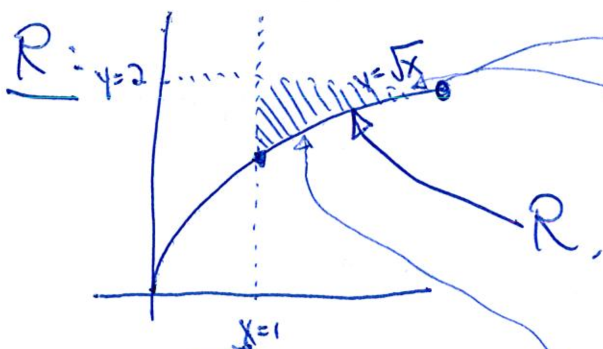
D. $\frac{dS}{dt} = k\frac{75}{S}$

E. $\frac{dS}{dt} = kS$

$\frac{dS}{dt} = k \cdot (75-S)$

Since S = amount dissolved, the amount that is undissolved is 75 (the amount we started with) minus S, i.e. $(75-S)$.

7. Consider the double integral $\iint_R e^{y^3}\, dA$, where R is the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 1$. Which of the following is an appropriate way to set up this double integral?



A. $\int_0^1 \int_{\sqrt{x}}^{2} e^{y^3}\, dy\, dx$

B. $\int_1^4 \int_2^{\sqrt{x}} e^{y^3}\, dy\, dx$

C. $\int_0^4 \int_0^{\sqrt{x}} e^{y^3}\, dy\, dx$

D. $\int_0^1 \int_2^{\sqrt{x}} e^{y^3}\, dy\, dx$

E. $\int_1^4 \int_{\sqrt{x}}^{2} e^{y^3}\, dy\, dx$

Y values start at the bottom, $y=\sqrt{x}$, then go up to the line $y=2$, So the limits for y are $\int_{\sqrt{x}}^{2} dy$.

The x values go from $x=1$ to the point where $y=\sqrt{x}$ and $y=2$ intersect, i.e. $2=\sqrt{x}$, so $x=4$. So $\int_1^4 dx$ are the limits on x.

8. Evaluate the double integral: $\int_0^2 \int_0^1 xy^2 e^{xy^3}\, dy dx$

A. $e^2 - 3$

B. $\frac{1}{3}e^2 + 1$

C. $\frac{1}{3}e^2 - 1$

D. $\frac{1}{3}e^2 - \frac{1}{3}$

E. $\frac{1}{3}e^2 + \frac{1}{3}$

Inner integral first: $\int_0^1 xy^2 e^{xy^3} dy$. Let $u = xy^3$.

Then $du = 3xy^2 dy$ so $dy = du/3xy^2$, so

$$\int_{y=0}^{y=1} xy^2 e^u \frac{du}{3xy^2} = \frac{1}{3}\int_{0=y}^{1=y} e^u du = \frac{1}{3}\left(e^u\Big|_{y=0}^{y=1}\right.$$

$$= \frac{1}{3}\left(e^{xy^3}\Big|_0^1\right. = \frac{1}{3}\left(e^x - e^0\right) = \frac{1}{3}(e^x - 1).$$ Outer integral: $\int_0^2 \frac{1}{3}(e^x - 1)\, dx$

$$= \frac{1}{3}\left(e^x - x\Big|_0^2\right. = \frac{1}{3}\left[(e^2 - 2) - (e^0 - 0)\right] = \frac{1}{3}(e^2 - 2 - 1) = \boxed{\frac{1}{3}e^2 - 1}$$

9. Find a power series for the function

$$f(x) = \frac{x^2}{1+x^2}$$

A. $\sum_{k=1}^{\infty}(-1)^{k-1}x^{2(k+2)}$

B. $\sum_{k=1}^{\infty}(-1)^{k-1}x^{2k}$

C. $\sum_{k=1}^{\infty}(-1)^{k+1}x^{2k+2}$

D. $\sum_{k=1}^{\infty}(-x)^{2k}$

E. $\sum_{k=1}^{\infty} -x^{2k+1}$

Looks like $\frac{a}{1-r}$. $\left(\frac{x^2}{1+x^2} = \frac{x^2}{1-(-x^2)}\right)$

So $a = x^2$, $r = -x^2$.

$\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$ so

$$\frac{x^2}{1-(-x^2)} = \sum_{n=0}^{\infty} x^2 \cdot (-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^2 x^{2n}$$

$= \sum_{n=0}^{\infty}(-1)^n x^{2+2n}$. But the answer options all start at $k=1$. One way to solve this is write out the first 2 terms for all the answer options and compare to your own. Another way is reindexing:

$$\sum_{n=0}^{\infty}(-1)^n x^{2+2n} \xrightarrow[n = k-1]{k=1 \text{ when } n=0 \text{ so } k=n+1} \sum_{k=1}^{\infty}(-1)^{k-1} x^{2+2(k-1)}$$ (replace n with k-1)

$$= \sum_{k=1}^{\infty}(-1)^{k-1} x^{2+2k-2} = \sum_{k=1}^{\infty}(-1)^{k-1} x^{2k}$$

10. Use the ratio test to determine whether each of the given series converges.

$$(i) \sum_{k=2}^{\infty} \frac{3^k}{k^2}; \quad (ii) \sum_{k=0}^{\infty} k\left(\frac{1}{5}\right)^k; \quad (iii) \sum_{k=3}^{\infty} \frac{6^k}{k!}$$

A. (iii) converges; (i) and (ii) diverge

B. (ii) and (iii) converge; (i) diverges

C. (i) and (ii) converge; (iii) diverges

D. (i) and (iii) converge; (ii) diverges

E. (i), (ii) and (iii) converge

(i) $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{3^{k+1}}{(k+1)^2}}{\left(\frac{3^k}{k^2}\right)}\right| = \left|\frac{3^{k+1} \cdot k^2}{3^k \cdot (k+1)^2}\right| = \left|3\frac{k^2}{(k+1)^2}\right|$. $\lim_{k\to\infty}\left|\frac{3k^2}{(k+1)^2}\right| = 3$

and $3>1$, so ratio test says (i) diverges.

(ii) $\sum_{k=0}^{\infty} k\left(\frac{1}{5}\right)^k$ has $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)\left(\frac{1}{5}\right)^{k+1}}{k\left(\frac{1}{5}\right)^k}\right| = \left|\frac{(k+1)}{k}\frac{1}{5}\right|$. $\lim_{k\to\infty}\left|\frac{(k+1)}{k}\left(\frac{1}{5}\right)\right| = \frac{1}{5}$

and $\frac{1}{5}<1$, so ratio test says (ii) converges. Already we can rule out answer options A, C, D, and E, so B must be the answer.

11. Evaluate the double integral $\int_6^7 \int_0^{\sqrt{x-6}} (x+2y)\,dy\,dx$

A. 38.4

B. 4.9

C. 0.5

D. 1

E. 0

Inner integral: $\int_0^{\sqrt{x-6}} (x+2y)\,dy = \left(xy + y^2\right)\Big|_0^{\sqrt{x-6}}$

$= \left(x(x-6)^{1/2} + \left((x-6)^{1/2}\right)^2\right) - (0)$

$= x(x-6)^{1/2} + x-6$

Outer integral: $\int_6^7 \left(x(x-6)^{1/2} + x-6\right)dx$. Let $u=x-6$, $du=dx$.

$= \int_{x=6}^{x=7} (x u^{1/2} + u)\,du$. To remove the last x, $u+6=x$] plug this in:

$\int_{x=6}^{x=7} \left((u+6)u^{1/2} + u\right)du = \int_{x=6}^{x=7}\left(u^{3/2} + 6u^{1/2} + u\right)du = \left(\frac{2}{5}u^{5/2} + \frac{2}{3}6u^{3/2} + \frac{1}{2}u^2\right|_{x=6}^{x=7}$

$= \left(\frac{2}{5}(x-6)^{5/2} + 4(x-6)^{3/2} + \frac{1}{2}(x-6)^2\right|_6^7 = \left(\frac{2}{5}\cdot 1\right) + (4\cdot 1) + \left(\frac{1}{2}\cdot 1\right) - (0+0+0)$

$= \frac{2}{5} + 4 + \frac{1}{2} = 4.9$

12. Use a Taylor polynomial of degree 4 to approximate: $\int_1^2 15e^{x^2}\,dx$. Round your answer to two decimal places.

A. 778.19

B. 389.10

C. 184.36

D. 163.97

E. 96.50

Find Taylor Series: e^{x^2} looks like e^x.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$.

Degree 4 Taylor Polynomial: list terms until the degree passes 4:

$n=0$: $\frac{x^0}{0!} = 1$ $n=1$: $\frac{x^{2\cdot 1}}{1!} = x^2$ $n=2$: $\frac{x^{2\cdot 2}}{2!} = \frac{x^4}{2}$ $n=3$: $\boxed{\frac{x^{2\cdot 3}}{3!} = \frac{x^6}{6}}$

degree too high.

So Taylor Poly $= 1 + x^2 + \frac{x^4}{2}$

$$\int_1^2 15e^{x^2}dx \approx 15\int_1^2\left(1+x^2+\frac{x^4}{2}\right)dx = 15\left(x + \frac{1}{3}x^3 + \frac{1}{10}x^5\right)\Big|_1^2$$

$$= 15\left[\left(2 + \frac{8}{3} + \frac{32}{10}\right) - \left(1 + \frac{1}{3} + \frac{1}{10}\right)\right] = 96.5$$

13. Suppose an ice cream factory puts ice cream into cylindrical containers. The volume of ice cream in each container is 300π cubic inches. What is the least amount of material required to make one container (include a simple lid for the top of the container)? Round your answer to the nearest 100 square inches.

A. 2200 square inches

B. 1100 square inches

C. 700 square inches

D. 500 square inches

E. 400 square inches

$V = 300\pi$ is given. Volume of a cylinder with radius r, height h, is $V = \text{base} \cdot h$

$V = \pi r^2 \cdot h$. Minimizing the material used is the same as minimizing the total surface area:

SA = base + top + side. base = top = πr^2. Side is $2\pi r h$ because

$SA = 2\pi r^2 + 2\pi r h$. Minimize $S(r,h) = 2\pi r^2 + 2\pi r h$ subject to the constraint $V(r,h) = \pi r^2 h = 300\pi$. Lagrange Equations: $4\pi r + 2\pi h = \lambda(2\pi r h)$

$2\pi r = \lambda \pi r^2$

$\pi r^2 h = 300\pi$

Divide both sides of $2\pi r = \lambda \pi r^2$ by πr^2:

$$\frac{2}{r} = \lambda$$ (we can divide by r

because $r \neq 0$, since it's given that $V = 300\pi = \pi r^2 h$).

Plug $\frac{2}{r} = \lambda$ into

$$4\pi r + 2\pi h = \lambda(2\pi r h)$$

$$4\pi r + 2\pi h = \frac{2}{r}(2\pi r h)$$

$$4\pi r + 2\pi h = 4\pi h \longrightarrow 4\pi r = 2\pi h$$

$2r = h$. Finally, plug this into the constraint equation:

$$\pi r^2 h = 300\pi \longrightarrow \pi r^2(2r) = 300\pi.$$

$$2\pi r^3 = 300\pi$$

$$r^3 = 150 \qquad r = (150)^{1/3}$$

and $h = 2 \cdot (150)^{1/3}$. Since this is the only critical point, it must be the minimum, so plug $(r, h) = (\sqrt[3]{150}, 2\sqrt[3]{150})$ into $S(r,h)$:

$$S(\sqrt[3]{150}, 2\sqrt[3]{150}) \approx \boxed{500 \text{ in}^2}$$

14. Suppose every year, Alistar makes a resolution to be more friendly. Accordingly, every January 1, he adds 2 people to his list of friends. However, remaining Alistar's friend is very difficult. The fraction of his friends which last for at least t years is

$$f(t) = 2^{-0.1t}$$

How many current friends will Alistar have on December 31 in the long run? (Round your answer to the nearest whole person.)

A. 16 friends

B. 28 friends

C. 32 friends

D. 14 friends

E. 56 friends

friends now = # friends remaining from 1 year ago
+ # " " 2 years ago
+ # " " 3 years ago
+ ...

$= 2\cdot f(1) + 2\cdot f(2) + 2\cdot f(3) + 2\cdot f(4) + \cdots$

$= 2\cdot 2^{-0.1\cdot 1} + 2\cdot 2^{-0.1\cdot 2} + 2\cdot 2^{-0.1\cdot 3} + 2\cdot 2^{-0.1\cdot 4} + \cdots$ A geometric series!

$a = 2\cdot 2^{-0.1\cdot 1} = 2^{1-0.1} = 2^{0.9}$. $r = (2\cdot 2^{-0.1\cdot 2}) / (2\cdot 2^{-0.1\cdot 1}) = 2^{-0.2+0.1} = 2^{-0.1}$

So total friends $= \frac{a}{1-r} = 2^{0.9} / (1 - 2^{-0.1}) \approx 28$ friends.

15. The temperature at a point (x, y) on a metal plate is $T(x, y) = xy$ degrees Celsius. An ant on the plate walks around the circle of radius $\sqrt{8}$ centered at the origin. What is the lowest temperature encountered by the ant?

A. -4 degrees Celsius

B. 4 degrees Celsius

C. 2 degrees Celsius

D. -2 degrees Celsius

E. $\sqrt{2}$ degrees Celsius

If the ant is on the edge of a circular plate the ant's position is given by the equation of a circle: $x^2+y^2=(\sqrt{8})^2=8$. So we want to minimize $T(x,y)=xy$ subject to the constraint $x^2+y^2=8$. Lagrange Equations:

$y = \lambda 2x$, $x = \lambda 2y$, $x^2+y^2=8$.

$\frac{y}{2x} = \lambda$ $\quad \frac{x}{2y} = \lambda$ So $\frac{y}{2x} = \frac{x}{2y} \longrightarrow 2y^2 = 2x^2 \to y^2 = x^2$

Plug this into $x^2+y^2=8$: $x^2+x^2=8 \to 2x^2=8 \to x^2=4 \to x=\pm 2$

Since $y^2=x^2$, $y=\pm 2$. Critical points to check: $(2,2)$ $(2,-2)$ $(-2,2)$ $(-2,-2)$

Lowest temperature at any of these points is $\boxed{-4}$.