SOLUTIONS

NOTE: My solution for a given problem is not necessarily the only correct way to do that problem; I accept any method of solving the problems that follows the instructions and uses material taught in this course. I give a lot of detail to make sure everything is clear—I don't expect you to give quite this much detail.

1 (2 points) Evaluate $\displaystyle\int_0^1 \int_{\ln 2}^{\ln 3} xe^{y-x}\, \mathrm{d}x\, \mathrm{d}y$

As always, first simpify! We can write $e^{y-x} = e^y e^{-x}$, so we can change this to be easier:

$$
\begin{aligned}
\int_0^1 \int_{\ln 2}^{\ln 3} xe^{y-x}\, \mathrm{d}x\, \mathrm{d}y &= \int_0^1 \int_{\ln 2}^{\ln 3} e^y xe^{-x}\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_0^1 e^y \int_{\ln 2}^{\ln 3} xe^{-x}\, \mathrm{d}x\, \mathrm{d}y \qquad \text{(since } y \text{ is a constant with respect to } x\text{)} \\
&= \int_0^1 e^y \int_{\ln 2}^{\ln 3} xe^{-x}\, \mathrm{d}x\, \mathrm{d}y
\end{aligned}
$$

Do the inside integral first: $\int xe^{-x}\, \mathrm{d}x$. It requires integration by parts. We want $\mathrm{d}v$ to be easy to integrate, but both x and e^{-x} are easy to integrate. We want our choice of u to have a simpler derivative. Only $u = x$ satisfies that, so we get $u = x$, $\mathrm{d}u = \mathrm{d}x$, $\mathrm{d}x = e^{-x}$ and $v = -e^{-x}$.

$$
\begin{aligned}
\int xe^{-x}\, \mathrm{d}x &= uv - \int v\, \mathrm{d}u \\
&= x(-e^{-x}) - \int (-e^{-x})\, \mathrm{d}x \qquad\qquad \text{(plugging in)} \\
&= -xe^{-x} + \int e^{-x}\, \mathrm{d}x \\
&= -xe^{-x} - e^{-x} = -e^{-x}(x+1)
\end{aligned}
$$

Now that we have this antiderivative, we can compute

$$
\begin{aligned}
\int_{\ln 2}^{\ln 3} xe^{-x}\, \mathrm{d}x &= \left(-e^{-x}(x+1)\right|_{\ln 2}^{\ln 3} \\
&= -\left[e^{-\ln 3}(1+\ln 3) - \left(e^{-\ln 2}(1+\ln 2)\right)\right] \\
&= -\left[\frac{1}{3}(1+\ln 3) - \frac{1}{2}(1+\ln 2)\right] \qquad\qquad \text{simplify}
\end{aligned}
$$

So now we have

$$\int_0^1 e^y \int_{\ln 2}^{\ln 3} xe^{-x} \, dx \, dy = -\int_0^1 e^y \left(\frac{1}{3}(1 + \ln 3) - \frac{1}{2}(1 + \ln 2) \right) dy \qquad \text{by plugging in}$$

$$= -\left(\frac{1}{3}(1 + \ln 3) - \frac{1}{2}(1 + \ln 2) \right) \int_0^1 e^y \, dy$$

$$= -\left(\frac{1}{3}(1 + \ln 3) - \frac{1}{2}(1 + \ln 2) \right) \left(e^y \big|_0^1 \right)$$

$$= -\left(\frac{1}{3}(1 + \ln 3) - \frac{1}{2}(1 + \ln 2) \right) \left(e^1 - e^0 \right)$$

$$= -\left(\frac{1}{3}(1 + \ln 3) - \frac{1}{2}(1 + \ln 2) \right) (e - 1)$$

$$= -\left(\frac{1}{3}\ln 3 - \frac{1}{6} - \frac{1}{2}\ln 2 \right) (e - 1) \qquad \text{final answer}$$

Don't round your answers unless the directions say that's ok, or the answer options on the exam are clearly rounded.

2 (3 points) Evaluate $\int_{-1}^1 \int_{x^{2/3}}^{x^2} x\sqrt{y} \, dy \, dx$ Begin by doing the inside integral:

$$\int_{x^{2/3}}^{x^2} x\sqrt{y} \, dy = \int_{x^{2/3}}^{x^2} xy^{1/2} \, dy$$

$$= x \left(\frac{2}{3} y^{3/2} \bigg|_{x^{2/3}}^{x^2} \right)$$

$$= x\frac{2}{3} \left((x^2)^{3/2} - (x^{3/2})^{3/2} \right) \qquad \text{plugging in}$$

$$= x\frac{2}{3} \left(x^3 - x \right) \qquad \text{simplifying}$$

$$= \frac{2}{3} \left(x^4 - x^2 \right) \qquad \text{simplifying more}$$

Now we do the outer integral:

$$
\int_{-1}^{1} \int_{x^{2/3}}^{x^2} x\sqrt{y} \, \mathrm{d}y \, \mathrm{d}x = \int_{-1}^{1} \frac{2}{3} \left(x^4 - x^2 \right) \, \mathrm{d}x
$$

$$
= \frac{2}{3} \left(\frac{x^5}{5} - \frac{x^3}{3} \right) \Big|_{-1}^{1}
$$

$$
= \frac{2}{3} \left[\left(\frac{1}{5} - \frac{1}{3} \right) - \left(\frac{-1}{5} - \frac{-1}{3} \right) \right]
$$

$$
= \frac{2}{3} \left[\frac{2}{5} - \frac{2}{3} \right]
$$

$$
= \frac{2}{3} \left[\frac{6}{15} - \frac{10}{15} \right]
$$

$$
= \frac{2}{3} \left[\frac{-4}{15} \right]
$$

$$
= \frac{-8}{45}
$$

3.1 (3 points) Set up the iterated integral for the following double integral:

$$
\iint_{R} 12xy \, \mathrm{d}A
$$

where R is the region bounded by the curves $x = y + 5$ and $x = y^2 - 1$.

If you draw these two curves, you can see that the parabola is on the left and the line is on the right of the bounded region. That means that for this region, we have $y^2 - 1 \leq x \leq y + 5$. So the inside integral looks like this:

$$
\int_{y^2 - 1}^{y+5} 12xy \, \mathrm{d}x
$$

To find the y limits of integration, we have to set our two curves equal to each other and then solve for y:

$$
y^2 - 1 = y + 5
$$

$$
y^2 - y - 6 = 0
$$

$$
(y - 3)(y + 2) = 0
$$

So $y = -2, 3$ are the y limits of integration. This gives us

$$
\int_{-2}^{3} \int_{y^2 - 1}^{y+5} 12xy \, \mathrm{d}x \, \mathrm{d}y
$$

3.2 (2 points) Evaluate the iterated integral that you set up in part 3.1.

We start with the inside integral:

$$\int_{y^2-1}^{y+5} 12xy \ \mathrm{d}x = 6y \left(x^2 \big|_{y^2-1}^{y+5}\right)$$

$$= 6y \left((y^2-1)^2 - (y+5)^2\right)$$
$$= 6y \left(y^4 - 2y^2 + 1 - (y^2 + 10y + 25)\right)$$
$$= 6y \left(y^4 - 3y^2 - 24\right)$$
$$= 6y^5 - 18y^3 - 144y$$

Now we do the outside integral:

$$\int_{-2}^{3} \int_{y^2-1}^{y+5} 12xy \ \mathrm{d}x \ \mathrm{d}y = \int_{-2}^{3} \left(6y^5 - 18y^3 - 144y\right) \ \mathrm{d}y$$

$$= \left(y^6 - \frac{9}{2}y^4 - 72y^2\right)\Big|_{-2}^{3}$$

$$= \left(3^6 - \frac{9}{2}3^4 - 72 \cdot 3^2\right) - \left((-2)^6 - \frac{9}{2}(-2)^4 - 72(-2)^2\right)$$

$$= \frac{1375}{2}$$