

## MA 224 - Quiz 11

### SOLUTIONS

NOTE: My solution for a given problem is not necessarily the only correct way to do that problem; I accept any method of solving the problems that follows the instructions and uses material taught in this course. I give a lot of detail to make sure everything is clear—I don't expect you to give quite this much detail.

1. (2 points) Write the following series in compact form (meaning write it using summation notation). Make sure your index starts at  $n = 0$ , and SHOW WORK.

$$1 - \frac{4}{1} + \frac{9}{2} - \frac{16}{6} + \frac{25}{24} - \dots$$

Look at the pattern that the numerator makes, and the pattern that the denominator makes.

Numerator: 1, 4, 9, 16, 25. These are the squares of 1, 2, 3, 4, and 5, so the numerator (starting at  $n = 0$ ) must be  $(n + 1)^2$

Denominator: 1, 1, 2, 6, 24. These are the first few factorials:  $1 = 0!$ ,  $1 = 1!$ ,  $2 = 2!$ ,  $6 = 3!$ ,  $24 = 4!$ , so the denominator (starting at  $n = 0$  must be  $n!$ )

Finally, there is an alternating minus sign, and it starts on positive: 1, -1, 1, -1, 1. Since we want to make sure  $n$  starts at 0, we can account for this by including the term  $(-1)^n$ .

So the series is 
$$\sum_{n=0}^{\infty} \frac{(-1)^n (n + 1)^2}{n!}$$

2. (2 points) Show whether or not this series converges (you don't have to say what number it converges to)

$$\sum_{n=2}^{\infty} \frac{(n - 1)^2}{3^{n+1}}$$

To show whether or not a series converges, we have to use the ratio test (unless it is a geometric series; then you use the geometric series test).

To do the ratio test, first we compute the limit  $L = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$ . To do that, first we'll simplify the

expression  $\frac{a_{k+1}}{a_k}$

$$\begin{aligned} &= \frac{((k+1)-1)^2}{3^{(k+1)+1}} \cdot \frac{3^{k+1}}{(k-1)^2} \\ &= \frac{k^2}{3^{k+2}} \cdot \frac{3^{k+1}}{(k-1)^2} \\ &= \frac{k^2 \cdot 3^{k+1}}{(k-1)^2 \cdot 3^{k+2}} \\ &= \frac{k^2 \cdot 3^{k+1}}{(k-1)^2 \cdot 3^{k+1} \cdot 3^1} \\ &= \frac{k^2}{(k-1)^2 \cdot 3} && \text{cancel the } 3^{k+1} \\ &= \frac{k^2}{3(k-1)^2} \end{aligned}$$

Now that we've simplified  $\frac{a_{k+1}}{a_k}$ , we can compute that limit.

$$\begin{aligned} L &= \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| \\ &= \lim_{k \rightarrow \infty} \left| \frac{k^2}{3(k-1)^2} \right| && \text{using our work above} \\ &= \frac{1}{3} \end{aligned}$$

the last step is because when you're taking the limit of a fraction with a polynomial in the numerator and denominator, and the degrees are equal, you just look at the ratio of the leading coefficients (in this case, the leading coefficient in the numerator is 1, and the leading coefficient in the denominator is 3).

Finally, since the limit is  $1/3 < 1$ , the ratio test says that the series converges.

3. (3 points) Find the interval of convergence for this power series:

$$\sum_{k=3}^{\infty} \frac{2^{k+1} x^k}{k!}$$

Again we use the ratio test, and again our first step is simplifying the expression  $\frac{a_{n+1}}{a_n}$ .

$$\begin{aligned}
 \frac{a_{n+1}}{a_n} &= \frac{2^{(n+1)+1}x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^{n+1}x^n} \\
 &= \frac{2^{n+1} \cdot 2^1 x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^{n+1}x^n} \\
 &= \frac{2^{n+1} \cdot 2^1 x^{n+1} n!}{2^{n+1} x^n (n+1)!} \\
 &= \frac{2^1 x^n x^1 n!}{x^n (n+1)!} && \text{cancel the } 2^{n+1} \\
 &= \frac{2xn!}{(n+1)!} && \text{cancel the } x^n \\
 &= \frac{2xn!}{(n+1) \cdot n!} && \text{rewrite } (n+1)! = (n+1) \cdot n! \\
 &= \frac{2x}{(n+1)} && \text{cancel the } n!'s
 \end{aligned}$$

At last we have the expression simplified, so we can take the limit of it:

$$\begin{aligned}
 L &= \lim_{n \rightarrow \infty} \left| \frac{2x}{(n+1)} \right| \\
 &= |2x| \lim_{n \rightarrow \infty} \frac{1}{(n+1)} && \text{factor out } |2x| \text{ since it's constant} \\
 &= 0 && \text{since the denominator is } \infty
 \end{aligned}$$

Since the limit is 0, that means the series converges, no matter what the value of  $x$  is. So the interval of convergence is  $(-\infty, \infty)$ .

4. (2 points) Find the interval of convergence for this power series:

$$\sum_{k=0}^{\infty} \frac{3^{k-1}x^k}{(k+1)^2}$$

Again, we want to use the ratio test, and once more our first step is simplifying the expression  $\frac{a_{n+1}}{a_n}$ .

$$\begin{aligned}
 \frac{a_{n+1}}{a_n} &= \frac{3^{(n+1)-1}x^{n+1}}{((n+1)+1)^2} \cdot \frac{(n+1)^2}{3^{n-1}x^n} \\
 &= \frac{3^n x^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{3^{n-1}x^n} \\
 &= \frac{3^n x^{n+1} (n+1)^2}{3^{n-1} x^n (n+2)^2} \\
 &= \frac{3^{n-1} \cdot 3 \cdot x^n \cdot x \cdot (n+1)^2}{3^{n-1} x^n (n+2)^2} && \text{since } x^{n+1} = x^n x \text{ and } 3^n = 3^{n-1} 3 \\
 &= \frac{3 \cdot x \cdot (n+1)^2}{(n+2)^2} && \text{cancel the } 3^{n-1} \text{ and } x^n
 \end{aligned}$$

Now that we simplified the expression, we can take its limit:

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{3x(n+1)^2}{(n+2)^2} \right| \\ &= |3x| \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(n+2)^2} \\ &= |3x|(1) && \text{the limit is 1 because the degrees are equal} \\ &= |3x| \end{aligned}$$

The ratio test says that the series converges only when the limit is  $< 1$ . But the limit is  $|3x|$ , so the series converges when  $|3x| < 1$ , i.e. when  $|x| < \frac{1}{3}$ . Hence, the interval of convergence is  $(-\frac{1}{3}, \frac{1}{3})$ .

5. (1 points) These are the first two terms of a geometric series. What does this series converge to? (SHOW how you get that number).

$$\frac{5}{6} - \frac{1}{2} + \dots$$

A geometric series always has the form  $a + ar + ar^2 + ar^3 + ar^4 + \dots$ . So  $a$  is just the first term, i.e.  $a = \frac{5}{6}$ . And we can find  $r$  by taking the ratio of the first two terms:  $\frac{ar}{a} = r$ . So for our geometric series  $r = \frac{-1/2}{5/6} = \frac{-1 \cdot 6}{2 \cdot 5} = \frac{-3}{5}$ .

Now that we have  $a$  and  $r$ , we can plug them in to the geometric series formula:  $\frac{a}{1-r} = \frac{\frac{5}{6}}{1-\frac{-3}{5}} = \frac{5/6}{1+3/5} = \frac{5/6}{8/5} = \frac{25}{48}$