MA 224 - Quiz 4

SOLUTIONS

NOTE: My solution for a given problem is not necessarily the only correct way to do that problem; I accept any method of solving the problems that follows the instructions and uses material taught in this course. I give a lot of detail to make sure everything is clear—I don't expect you to give quite this much detail.

1. (10 pts) Find the area of the region bounded by the curves

$$y = x^3 - 5x + 5$$
$$y = x + 5 - x^2$$

To find the area between two curves, we use the formula

$$\text{area} = \int_a^b [f(x) - g(x)]\ \mathrm{d}x$$

where $f(x)$ and $g(x)$ are the functions giving us the two curves, and $f(x)$ is the top function on the interval $a \le x \le b$.

If the problem does not specify the interval $a \le x \le b$, then we have to find it. We do this by finding where the two curves, $f(x)$ and $g(x)$, intersect each other; we find where they intersect by setting them equal to each other, $f(x) = g(x)$, and then solving for x. Since we don't want to spend extra time figuring out which function is actually on top, just ignore that aspect of the problem for now! So it doesn't matter which function we pick to be $f(x)$ and which to be $g(x)$.

$$\begin{aligned} f(x) &= g(x) \\ x^3 - 5x + 5 &= x + 5 - x^2 \\ x^3 + x^2 - 6x &= 0 && \text{simplify and collect like terms} \\ x(x^2 + x - 6) &= 0 && \text{factor out } x \\ x(x+3)(x-2) &= 0 && \text{factor the rest} \end{aligned}$$

So now we know $x = -3, 0, 2$ are the x-values where the two curves intersect. Since the curves intersect more than 2 times, this means they might be crossing each other and creating areas both above AND below the x-axis—this means the areas might cancel out with each other if we are not careful. To avoid this kind of area cancellation, we have to break the integral $\int_a^b [f(x) - g(x)]\ \mathrm{d}x$, into multiple pieces:

one piece over the interval $-3 \le x \le 0$ and the other piece over the interval $0 \le x \le 2$. So we get

$$\text{area} = \left| \int_{-3}^{0} [f(x) - g(x)]\ \mathrm{d}x \right| + \left| \int_{0}^{2} [f(x) - g(x)]\ \mathrm{d}x \right|$$

NOTE: we take the absolute value of each of the two integrals because we want the *area* in that region, and area is always positive. We have to take the absolute value here because we were lazy earlier and skipped the step of figuring out which function was on top, $f(x)$ or $g(x)$. If this sounds complicated, don't worry about it—just take the positive version of whatever number you get when you compute these two definite integrals.

EXAM TIP: Now we have to do each of these integrals. Note that we already figured out what $[f(x) - g(x)]$ is in our earlier work when we set $f(x) = g(x)$. We figured out that $[f(x) - g(x)] = x^3 + x^2 - 6x$, so you can avoid doing that work a second time.

NEXT TIP: The next step in computing the integral

$$\int_{-3}^{0} [f(x) - g(x)] \, dx = \int_{-3}^{0} \left[x^3 + x^2 - 6x\right] \, dx$$

is for us to find an antiderivative of the integrand, $(x^3 + x^2 - 6x)$. We do that using the power rule a few times:

$\int (x^3 + x^2 - 6x) \, dx = \frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2 + C$. Since we just want one antiderivative, not the whole family of antiderivatives, we just omit the $+C$ and take $\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2$.

Here's the tip: this antiderivative works for both of our two integral pieces: for both

$$\int_{-3}^{0} [f(x) - g(x)] \, dx$$

and

$$\int_{0}^{2} [f(x) - g(x)] \, dx$$

we'll use the function $F(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2$ as the antiderivative in BOTH pieces:

$$\int_{-3}^{0} \left[x^3 + x^2 - 6x\right] \, dx = \left(F(x)\right|_{-3}^{0} = \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2\right|_{-3}^{0} \tag{1}$$

$$\int_{0}^{2} \left[x^3 + x^2 - 6x\right] \, dx = \left(F(x)\right|_{0}^{2} = \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2\right|_{0}^{2} \tag{2}$$

FINAL TIP: You can save more time by plugging in the number 0 just one time. Since you have to evaluate $F(0)$ for both of the above definite integrals, just evaluate it one time and use that result in both definite integrals!

Computing (1): We plug in to get

$$\begin{aligned} F(0) - F(-3) &= 0 - \left[\frac{(-3)^4}{4} + \frac{(-3)^3}{3} - 3(-3)^2\right] && \text{careful to distribute!} \\ &= -\frac{81}{4} + 9 + 27 \\ &= \frac{63}{4} && \text{for integral (1)} \end{aligned}$$

Computing (2): We plug in to get

$$\begin{aligned} F(2) - F(0) &= \left[\frac{(2)^4}{4} + \frac{(2)^3}{3} - 3(2)^2\right] - 0 && \text{reuse } F(0) = 0 \\ &= 4 + \frac{8}{3} - 12 \\ &= \frac{-16}{3} && \text{for integral (2)} \end{aligned}$$

At last we take the positive versions of each of these numbers and add them together. The total area between our two curves is $\frac{63}{4} + \frac{16}{3} = \frac{253}{12}$