MA 224 - Quiz 6

SOLUTIONS

NOTE: My solution for a given problem is not necessarily the only correct way to do that problem; I accept any method of solving the problems that follows the instructions and uses material taught in this course. I give a lot of detail to make sure everything is clear—I don't expect you to give quite this much detail.

1. (2 pts) Evaluate the limit

$$\lim_{N\to\infty} \frac{10}{\ln(N)}$$

Since $\ln(N) \to \infty$ as $N \to \infty$, that means the denominator of this goes to infinity while the numerator stays constant. Dividing by a giant number makes your expression very very small—so this limit is 0. Conceptually, think of it as "a constant divided by infinity is 0".

2. (2 pts) Evaluate the limit

$$\lim_{N\to\infty} \frac{\ln(N^2)}{N}$$

If we rewrite $\ln(N^2)$ as $2\ln(N)$ using log properties, then we can rewrite this as $\lim_{N\to\infty} \frac{\ln(N^2)}{N} = 2 \cdot \lim_{N\to\infty} \frac{\ln(N)}{N}$. Then, since we know from class that $\lim_{N\to\infty} \frac{\ln(N)}{N} = 0$, we have that the answer is 0.

3. (6 pts) Compute:

$$\int_0^\infty \frac{x^2}{1+x^3}\,\mathrm{d}x$$

With any improper integral, you must first rewrite it as a limit of a proper integral:

$$\int_0^\infty \frac{x^2}{1+x^3}\,\mathrm{d}x = \lim_{N\to\infty} \int_0^N \frac{x^2}{1+x^3}\,\mathrm{d}x$$

Then, we evaluate the definite integral. To do that, we need an antiderivative of $\frac{x^2}{1+x^3}$, so we do the integral $\int \frac{x^2}{1+x^3}\,\mathrm{d}x$.

Any time you have a nasty looking denominator, there's a good chance you want a u-substitution with $u =$ the denominator. If we try that here, we get $u = 1 + x^3$ and $\mathrm{d}u = 3x^2\,\mathrm{d}x$. Solving for $\mathrm{d}x$ gives $\mathrm{d}x = \frac{\mathrm{d}u}{3x^2}$. Finally, we can plug in:

$$\begin{aligned}
\int \frac{x^2}{1+x^3}\,\mathrm{d}x &= \int \frac{x^2}{u}\frac{\mathrm{d}u}{3x^2} \\
&= \frac{1}{3}\cdot\int \frac{1}{u}\,\mathrm{d}u && \text{cancelling the } x^2\text{'s} \\
&= \frac{1}{3}\ln|u| + C \\
&= \frac{1}{3}\ln|1+x^3| + C
\end{aligned}$$

Now we can return to the definite integral:

$$\begin{aligned}
\int_0^N \frac{x^2}{1+x^3}\,\mathrm{d}x &= \left(\frac{1}{3}\ln|1+x^3|\right|_0^N \\
&= \left(\frac{\ln|1+N^3|}{3} - \frac{\ln|1+0|}{3}\right) \\
&= \frac{\ln|1+N^3|}{3} && \text{since } \ln|1| = \ln(1) = 0
\end{aligned}$$

And at last we can compute the limit:

$$\begin{aligned}
\int_0^\infty \frac{x^2}{1+x^3}\,\mathrm{d}x &= \lim_{N\to\infty}\int_0^N \frac{x^2}{1+x^3}\,\mathrm{d}x \\
&= \lim_{N\to\infty}\frac{\ln|1+N^3|}{3} \\
&= \infty && \text{since } \ln(N) \to \infty \text{ as } N \to +\infty
\end{aligned}$$