Linear Independence, Span, and Basis of a Set of Vectors

What is linear independence?

A set of vectors \( S = \{v_1, \ldots, v_k\} \) is **linearly independent** if none of the vectors \( v_i \) can be written as a linear combination of the other vectors, i.e. \( v_j = \alpha_1 v_1 + \cdots + \alpha_k v_k \).

Suppose the vector \( v_j \) can be written as a linear combination of the other vectors, i.e. there exist scalars \( \alpha_i \) such that \( v_j = \alpha_1 v_1 + \cdots + \alpha_k v_k \) holds. (This is equivalent to saying that the vectors \( v_1, \ldots, v_k \) are linearly dependent).

We can subtract \( v_j \) to move it over to the other side to get an expression \( 0 = \alpha_1 v_1 + \cdots + \alpha_k v_k \) (where the term \( v_j \) now appears on the right hand side).

In other words, the condition that “the set of vectors \( S = \{v_1, \ldots, v_k\} \) is linearly dependent” is equivalent to the condition that there exists \( \alpha_i \) not all of which are zero such that

\[
0 = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}.
\]

More concisely, form the matrix \( V \) whose columns are the vectors \( v_i \). Then the set \( S \) of vectors \( v_i \) is a linearly dependent set if there is a nonzero solution \( x \) such that \( Vx = 0 \).

This means that the condition that “the set of vectors \( S = \{v_1, \ldots, v_k\} \) is linearly independent” is equivalent to the condition that there exists \( \alpha_i \) not all of which are zero such that

\[
0 = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}.
\]

How do you determine if a set is lin. ind.?

To determine if a set \( S = \{v_1, \ldots, v_k\} \) is linearly independent, we have to determine if the equation \( Vx = 0 \) has solutions other than \( x = 0 \). To do this,

1. Form the matrix \( V \) whose columns are the vectors \( v_i \).
2. Put \( V \) in row echelon form. Denote the row echelon form of \( V \) by \( \text{ref}(V) \)
3. check if each column contains a leading 1.

If every column of \( \text{ref}(V) \) contains a leading 1, then \( S = \{v_1, \ldots, v_k\} \) is **linearly independent**. Otherwise, the set \( S \) is linearly **dependent**.

Example: Let \( V = \mathbb{R}^4 \), and let \( T = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & \frac{1}{2} \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \). Is \( T \) linearly independent?
To answer this, we do the following:

1. Form a matrix whose columns are the vectors in $T$. Call the matrix $M_T$.
2. Row reduce $T$ until it is in row echelon form, $\text{ref}(M_T)$.
3. Check if each column has a leading 1.

**Step 1.** Form a matrix $M_T$ whose columns are the vectors in the set $T$:

\[
\begin{bmatrix}
1 & 3 & -2 \\
0 & 1 & 1 \\
2 & 0 & 2 \\
0 & 1 & -1
\end{bmatrix} \rightarrow M_T = \begin{bmatrix}
1 & 3 & -2 \\
0 & 1 & 1 \\
2 & 0 & 2 \\
0 & 1 & -1
\end{bmatrix}
\]

**Step 2.** Row reduce the matrix $M_T$.

\[
\begin{bmatrix}
1 & 3 & -2 \\
0 & 1 & 1 \\
2 & 0 & 2 \\
0 & 1 & -1
\end{bmatrix} \rightarrow R_3 \rightarrow R_3 - 2R_1 \rightarrow \begin{bmatrix}
1 & 3 & -2 \\
0 & 1 & 1 \\
0 & -6 & 6 \\
0 & 1 & -1
\end{bmatrix} \rightarrow R_3 \rightarrow R_3 + 6R_4 \rightarrow \begin{bmatrix}
1 & 3 & -2 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 1 & -1
\end{bmatrix} \rightarrow R_3 \leftrightarrow R_4 \rightarrow \begin{bmatrix}
1 & 3 & -2 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \rightarrow R_3 \rightarrow R_3 - R_2 \rightarrow \begin{bmatrix}
1 & 3 & -2 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \rightarrow R_3 \rightarrow \frac{1}{2}R_3 \rightarrow \begin{bmatrix}
1 & 3 & -2 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and now we can stop because we’ve reached row echelon form.

**Step 3.** What does this tell us? Because the row echelon form has a “leading 1” in each column, the columns of the original matrix are linear independent. This also tells us the vectors in our original set $T$ are also linearly independent.

On the other hand, if any columns of the row echelon form did not contain a leading 1, then the original column vectors would then be linear dependent.

### Determining if a set of vectors spans a vectorspace

A set of vectors $F = \{f_1, \cdots, f_n\}$ taken from a vectorspace $V$ is said to span the vectorspace if every vector in the vectorspace $V$ can be expressed as a linear combination of the elements in $F$. In other words, every vector $x$ in $V$ can be written $x = y_1f_1 + \cdots + y_nf_n$ for some scalars $y_j$. We can rewrite this idea from a matrix perspective:

\[
x = y_1f_1 + \cdots + y_nf_n = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \tag{1}
\]

This matrix approach leads us to the method we use to determine whether our set of vectors $F$ spans the vectorspace $V$. Let’s be more concrete. The vectorspaces we deal with in this class tend to be like $\mathbb{R}^n$ – the set of vectors with $n$ entries that are any real numbers. To show that the set $F$ spans the vectorspace $\mathbb{R}^n$, we do the following:

0. Form the matrix $F = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}$ with the vectors $f_j$ as its columns
1. Compute the reduced row echelon form of that matrix $F$, $\text{ref}(F)$.
2. If \( \text{rref}(F) \) has a leading 1 in every row, then the set \( F \) spans the vectorspace \( \mathbb{R}^n \)!

### Determining if a set of vectors is a basis for a vectorspace

A **basis** for a vectorspace \( V \) is a set of vectors \( B = \{b_1, \cdots, b_m\} \) that (1) span the vectorspace \( B \); and (2) are linearly independent.

To determine if a set \( B = \{b_1, \cdots, b_m\} \) of vectors spans \( V \), do the following:

0. Form the matrix \( B = [b_1 \; \cdots \; b_m] \)
1. Compute \( \text{rref}(B) \)
2. Test for linear independence: does every column of \( \text{rref}(B) \) have a leading 1? (if yes, the set \( B \) is linearly independent)
3. Test whether \( B \) spans the vectorspace: does every row of \( \text{rref}(B) \) have a leading 1? (If yes, then the set \( B \) spans the vectorspace).
4. If \( B \) passes both tests, then the set \( B \) is a basis!

### Determining a linearly independent subset of a set of vectors

Suppose we find out that the set of vectors \( G = \{g_1, \cdots, g_k\} \) spans the vectorspace \( \mathbb{R}^m \), but the set \( G \) is not linearly independent. How can we find a subset of \( G \) that is linearly independent? In other words, can we find a basis for our vectorspace \( \mathbb{R}^m \) hidden inside our linearly dependent set of vectors \( G \)?

Do the following:

0. As always, first form a matrix \( G = [g_1 \; \cdots \; g_k] \)
1. Then compute \( \text{rref}(G) \).
2. Each column of \( \text{rref}(G) \) that contains a leading 1 corresponds to a vector \( g_j \) in the original set \( G \). Let \( S \) be the subset of those vectors. Then \( S \) is linearly independent, AND \( \text{span}(S) = \text{span}(G) \). This means that \( S \) is a basis for the span of \( G \)!!

**Example:** Let \( V = \mathbb{R}^3 \), and let \( W = \left\{ \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ 6 \end{bmatrix} \right\} \). Find a subset of \( W \) that is a basis for \( V \).

**Step 0.** First form the matrix \( W = \begin{bmatrix} 2 & 2 & 0 & -2 \\ 4 & 1 & 1 & 6 \\ -4 & -2 & 8 & 6 \end{bmatrix} \).

**Step 1.** Compute \( \text{rref}(W) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \).
Step 2. First note that not every column contains a leading 1 – that means that our original set $T$ is not linearly independent, and so it cannot be a basis. However, the first three columns of rref($W$) contain a leading 1. If we let $S = \{w_1, w_2, w_3\}$ (since those are the three vectors that correspond to the columns of rref($W$) that contain leading 1s), then $S$ is a linearly independent set. Since each row of rref($W$) contains a leading 1, we know that $W$ spans the vectorspace. But the columns of rref($W$) that correspond to our subset of vectors, $S$, also all contain leading 1s (our subset $S$ is the first three vectors, $w_1, w_2, w_3$; this corresponds to the first three columns of rref($W$)) – that means that our subset $S$ still spans the vectorspace!

1 Determining a basis for span($S$) without using vectors from $S$

We have seen already that you can locate a linearly independent set of vectors within the set of vectors $S = \{s_1, \ldots, s_m\}$ by forming a matrix $S = [s_1 \cdots s_m]$, computing rref($S$), and then taking each of the vectors $s_j$ that corresponds to a column of rref($S$) that contains a leading 1.

In lessons 22-23 (class on 10/20, 10/22) we’ll look at an examples of finding a basis for $S$ using things other than vectors taken directly from $S$. 