11) Let T be the linear operator on F^2 which is represented in the standard ordered basis by the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Let $a_1 = (0, 1)$. Show that $F^2 \neq Z(a_1, T)$ and that there is no non-zero vector a_2 in F^2 such that $Z(a_2, T)$ is disjoint from $Z(a_1, T)$.

11) Let T be a linear operator on V an n-dimensional vectorspace and let $R = T(V)$ be the range of T.
(a) Prove that R has a complementary T-invariant subspace iff R is independent of $N = \operatorname{null} T$.
(b) If R and N are independent, prove that N is the unique T-invariant subspace complementary to R.

11) Let T be the linear operator on $\mathbb{R}^3$ which is represented by the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Let W be the null space of $T - 2I$. Prove that W has no complementary T-invariant subspace.

11) Let T be the linear operator on F^4 which is represented by the matrix $\begin{bmatrix} c & 0 & 0 & 0 \\ 1 & c & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & c \end{bmatrix}$ and let W be the nullspace of $T - cI$.
(a) Prove that W is the subspace spanned by $e_4 = (1, 0, 0, 0)$.
(b) Find the monic generators of the ideals $S(e_4; W), S(e_3; W), S(e_2; W), S(e_1; W)$ where $S(v, W)$ is the T-conductor of v into W, i.e. the ideal of polynomials $g(x)$ such that $g(T)v \in W$.

11) Let T be a linear operator over a subfield of $\mathbb{C}$ with matrix representation $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & a & 2 & 0 \\ 0 & 0 & b & 2 \end{bmatrix}$. Find the characteristic polynomial of T. Find the minimal polynomial of T and vectors satisfying theorem 3 in each of these cases: $a = 1 = b$; $a = 0 = b$; $a = 0$, $b = 1$

11) For $A, B \in F^{3\times3}$, show that $A\ B$ iff the characteristic and minimal polynomials of A are the same as those of B.

11) Let F be a subfield of $\mathbb{C}$ and let A and B be $n \times n$ matrices over F. Prove that if A and B are similar over $\mathbb{C}$ then they are similar over F.

11) Let A be an $n \times n$ matrix over $\mathbb{C}$. Prove that if every characteristic value of A is real then A is similar to a matrix with real entries.

11) Let T be a linear operator on the finite-dimensional vectorspace V. Prove that there exists a vector v in V such that if f is a polynomial and $f(T)v = 0$ then $f(T) = 0$ (this is called a separating vector). When T has a cyclic vector give a direct proof that any cyclic vector is a separating vector for T.

11) Let F be a subfield of $\mathbb{C}$ and let A be an $n \times n$ matrix of F. Let p be the minimal polynomial for A. If we regard A as a matrix over $\mathbb{C}$, then A has a minimal polynomial f as an $n \times n$ matrix over $\mathbb{C}$. Show that $p = f$ (using "a theorem on linear equations"). Can you prove it using the cyclic decomposition theorem?

11) Let T be a linear operator on an n-dimensional vectorspace V over F. Show that if the minimal polynomial for T is a power of an irreducible polynomial and the minimal polynomial is equal to the characteristic polynomial then no non-trivial T-invariant subspace has a complementary T-invariant subspace.

11) Show that if T is a diagonalizable linear operator on V then every T-invariant subspace of V has a complementary T-invariant subspace.

11) Let T be a linear operator on the n-dimensional vectorspace V. Prove that T has a cyclic vector iff every linear operator U which commutes with T is a polynomial in T.

11) Let V be an n-dimensional vectorspace over the field F and let T be a linear operator on V. Prove that every nonzero vector in V is a cyclic vector for T iff the characteristic polynomial for T is irreducible over F.

11) Let A be an $n \times n$ matrix over $\mathbb{R}$. Let T be the linear operator on $\mathbb{R}^n$ represented by A, and let U be the linear operator on $\mathbb{C}^n$ which is represented by A. Use 20 to prove that if the only subspaces invariant under T are $\mathbb{R}^n$ and 0 then U is diagonalizable.