3) Let V be an n dimensional vectorspace. Let $T : V \to V$ be a linear tranformation such that $T(V) = \operatorname{null} T$. Show n is even.

3) Let V be a vectorspace and $T : V \to V$ a linear transformation. Show that "$T(V) \cap \operatorname{null} T = \{0\}$" $\iff$ "$T(T\alpha) = 0 \Rightarrow T\alpha = 0$".

3) Let V be a vectorspace and T a linear operator on V. If $T^2 = 0$ but $T \neq 0$ what can we say about the relationship of the range of T to the nullspace of T?

3) Let V be a vectorspace with $\dim V = n$ and let T be a linear operator on V such that $\operatorname{rank} T = \operatorname{rank} T^2$. Show that $T(V) \cap \operatorname{null} T = \{0\}$.

3) Let $T \in L(F^n, F^n)$, let A be the matrix of T in the standard ordered basis for F^n, and let W be the subspace spanned by the column vectors of A.

3) Find a basis for the range and the null space of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

3) $T(x, y) = (-y, x)$

(a) The matrix for T in the standard ordered basis for $\mathbb{R}^2$ is given by

$Te_1 = (0, 1) = 0e_1 + 1e_2; Te_2 = (-1, 0) = (-1)e_1 + 0e_2$, i.e.$[T]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) Using the new ordered basis $B' = \{(1, 2); (1, -1)\}$ we have

$Te'_1 = (2, 1) = 1e'_1 + 1e'_2; Te'_2 = (-1, 1) = 0e'_1 + (-1)e'_2$, i.e.$[T]_{B'} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

(c) Prove that for every real number c the linear operator $(T - cI)$ is invertible.

$(T - cI)(x, y) = T(x, y) - c(x, y) = (-y, x) - (cx, cy) = (-cx - y, x - cy)$ To compute $[(T - cI)]_B$, $(T - cI)e_1 = (-c, 1) = (-c)e_1 + 1e_2; (T - cI)e_2 = (-1, -c) = (-1)e_1 + (-c)e_2$, i.e.$[(T - cI)]_B = \begin{bmatrix} -c & 1 \\ -1 & -c \end{bmatrix}$. But the matrix $M = \frac{1}{c^2+1} \begin{bmatrix} -c & -1 \\ 1 & -c \end{bmatrix}$ can be seen to be the inverse of $[(T - cI)]_B$ by multiplying them out, so the linear operator corresponding to M must be the inverse of I. We know there exists a linear operator corresponding the the matrix M by theorem 11 from chapter 3. Since M is a left and right inverse to $[(T - cI)]_B$ we see that the corresponding linear operator is also the left and right inverse of $T - cI$, and so $T - cI$ is an invertible linear operator.

(d) Let $B' = b_1, b_2$ be any ordered basis for $\mathbb{R}^2$ and let $[T]_{B'} = A$. Show $A_{12}A_{21} \neq 0$.

Suppose by way of contradiction that $A_{12}A_{21} \neq 0$. By theorem 14 from ch 3 we know that there exists some invertible matrix P such that $[T]_B = P^{-1}AP$ so we have $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = P^{-1} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} P$. We showed in part c that $T - cI = [T]_B - [cI]_B = [T]_B - cI$ is invertible for any c. So subtracting cI from both sides of the above equation gives $[T]_B - cI = P^{-1}AP - cI$ where the left side is invertible. Multiplying both sides on the left by P and the right by P^{-1} gives $P([T]_B - cI)P^{-1} = A - P(cI)P^{-1} = A - c(PIP^{-1}) = A - cI$. Since each of $P, P^{-1}, [T]_B - cI$ is invertible we know their product is invertible. Hence, the right side must be invertible, for any choice of c. But if we let $c = A_22$, then $A - cI = \begin{bmatrix} A_{11} - A_{22} & A_{12} \\ A_{21} & 0 \end{bmatrix}$. If $A_{12}A_{21} \neq 0$ then at least one of them is equal to 0, and so either the bottom row or the right column of the matrix $A - cI$ is all zeros, and hence the matrix cannot be invertible. This is a contradiction, and hence $A_{12}A_{21} \neq 0$.

10 Let S be a linear operator on $\mathbb{R}^2$ satisfying $S^2 = S$. Show that either $S = I$, $S = 0$, or an ordered basis B for $\mathbb{R}^2$ exists for which the matrix representation M of S relative to B is given by $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, i.e. $[S]_B = A$.

Suppose S is invertible. Then $S^{-1}S^2 = S^{-1}S$ because $S^2 = S$, which gives $S = I$. Now assume that S is not invertible. Since S is not invertible, its matrix representation M (relative to any basis) is not invertible, and therefore has rank ≤ 1. If $\operatorname{rank} A = 0$ then A is the 0 matrix and so S would be the zero linear operator.

BETTER SOLUTION: If S is not the zero linear operator, it has rank 1. Thus, $S(V)$ has rank 1 and (S) has rank 1, so take the basis elements from each of those dimension 1 spaces. Together they form a basis for $\mathbb{R}^2$ because neither one is a multiple of the other, or else they would belong to each other's spans (contradicting the range and the nullspace being distinct). Hence, $S(n_1) = 0n_1 + 0b_1$ and $S(b_1) = (1)b_1 + 0(n_1)$.

Thus, we can assume that S is not the identity or zero linear operator (and so M is not the 0 or identity matrix). Hence, M must have $\operatorname{rank} M > 0$, i.e. $\operatorname{rank} M = 1$. This means that there exists a linear dependency among the columns of M (since the row rank, column rank, and rank of the matrix are all the same), i.e. $k_1C_1 + k_2C_2 = 0$ for some nonzero scalars k_1, k_2 and the column vectors of A, C_1 and C_2. But this dependency implies $C_1 = k_1^{-1}k_2C_2$ (since the scalars are nonzero we can divide), i.e. one column is a multiple of the other column. A similar proof shows that one row is a multiple of the other. So we can write M (relative to the standard basis) as $M = \begin{bmatrix} a & ak \\ b & bk \end{bmatrix}$. Now since $M^2 = M$, we know from multiplying the matrices out and comparing the first entry in each that we must have $a = a^2 + abk = a(a + bk)$. If we assume a is nonzero then dividing gives us $1 = a + bk$, i.e. $1 - a = bk$. We'll use this fact in a little bit, and we'll return to the case $a = 0$ later.

For now, consider the basis $B = \{\beta_1 = (ka, 1 - a), \beta_2 = (-k, 1)\}$. Row reducing the matrix $\begin{bmatrix} -k & 1 \\ ak & 1-a \end{bmatrix}$ gives I, so we know that these vectors are a basis. Now, note that
$\begin{bmatrix} a & ak \\ b & bk \end{bmatrix}(ka, 1 - a) = (ka, kb)$ but we showed above that $kb = 1 - a$, so we have $M\beta_1 = 1\beta_1 + 0\beta_2$. Similarly, $\begin{bmatrix} a & ak \\ b & bk \end{bmatrix}(-k, 1) = (0, 0)$, i.e. $M\beta_2 = 0\beta_1 + 0\beta_2$.

This shows that the matrix representing S relative to the basis B is $[S]_B \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$ as desired.

Now we return to the case $a = 0$. Looking again at the computation $M^2 = M = \begin{bmatrix} a & ak \\ b & bk \end{bmatrix}$ we see that $b = ab + b^2k = b(a + bk)$. If b is nonzero then by dividing we again have $1 - a = bk$. The proof above can proceed from here. If b is also zero, then we have $a = 0$ and $b = 0$. Since we've assumed that M is not the 0 matrix, one of the entries in the right column is nonzero, and we should have labelled the matrix $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$. With at least one of a or b nonzero. Again computing $M^2 = M$ gives us the equations $ab = a$ and $b^2 = b$. Hence b must be 0 or 1. If b is 0 then the first equation implies a is zero, giving the 0 matrix, a contradiction. So $b = 1$, and a can be anything. If a is 0 then $B = \{\beta_1 = (0, 1), \beta_2 = (1, 0)\}$ gives $[S]_B = A$ as above and we're done.

If, on the other hand, a is nonzero, then $\beta_1 = (a, 1), \beta_2 = (1, 0)$ gives $[S]_B = A$ as above and we're done. This completes all cases.

12 For V a vectorspace with ordered basis $B = \{a_1, \ldots, a_n\}$ define a linear operator (via theorem 1) such that $Ta_1 = a_2, \cdots, Ta_i = a_{i+1}, \cdots, Ta_{n-1} = a_n$ and $Ta_n = 0$.

(a) Find the matrix A corresponding to T in basis B.

(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

(c) Let S be a linear operator on V satisfying $S^n = 0$ but $S^{n-1} \neq 0$. Prove there exists an ordered basis B' for V such that $[S]_{B'} = A$ the matrix described in part (a).

(d) Let M and N be any $n \times n$ matrices over F such that $M^n = N^n = 0$ but $M^{n-1} \neq 0 \neq N^{n-1}$. Show M and N are similar.