9) Given a linear operator T has matrix representaiton $A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$, express its minimal polynomial as the product of monic irreducible polynomials, $p_1 p_2$. Simple computations confirm that the minimal polynomial for T is $(X - 2I)(X^2 + I)$, so $p_1(X) = X - 2I$, $p_2(X) = X^2 + I$.

9) Given a linear operator T has matrix representaiton $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$, show that there is a diagonalizable operator D and a nilpotent operator N on $\mathbb{R}^3$ such that $T = D + N$ and $ND = DN$. Find the matrices for D and N.

9) Let V be the vectorspace of all polynomials of degree $\leq n$ over a field F. Show that the differentiation operator is nilpotent.

9) Let T be a linear operator on a finite dimensional vectorspace V with minimal polynomial $f = (x-c_1)^{d_1} \cdots (x-c_k)^{d_k}$. Let W_i be the null space of $(T - c_i I)^{d_i}$.

(a) Show that the set S of vectors b such that there exists some m for which $(T - c_i I)^m b = 0$ is equal to the subspace W_i.

(b) Show that the dimension of W_i is d_i.

9) Let V be a finite dimensional vectorspace over $\mathbb{C}$. Let T be a lin op on $\mathbb{C}$ and let D be the diagonalizable part of T. Prove for any $p(x) \in \mathbb{C}[x]$ that the diagonalizable part of $p(T)$ is $p(D)$.

9) Let T be a linear operator on V a finite dimensional vectorspace over F such that $\operatorname{rank} T \geq 1$. Show that it can't be the case that T is both diagonalizable and nilpotent.

9) Let V be a finite dimensional vector space over F and let T be a linear operator on V such that T commutes with every diagonalizable linear operator on V. Show T is a scalar matrix.

9) Let V be the space of $n \times n$ over F and let A be a fixed $n \times n$ matrix. Define T by $T(B) = AB - BA$ for all $B \in F^{n \times n}$. Show that T is nilpotent if A is.

9 Find two 4×4 nilpotent matrices such that they have the same minimal polynomial, but they are not similar.

9) T is a linear operator on a finite dimensional vectorspace V and has minimal polynomial $p = p_1^{r_1} \cdots p_k^{r_k}$. Let $V = W_1 \oplus \cdots \oplus W_k$ where W_i is the null space of $p_i(T)^{r_k}$, and let W be any subspace of V that is T-invariant. Show that $W = (W \cap W_1) \oplus \cdots \oplus (W \cap W_k)$.

9) Find the flaw in the proof: the minimal polynomial for T factors into linear factors, so it's triangulable. So there exists ordered basis B such that $A = [T]_B$ is upper triangular, so we can write it $A = D + N$ where D is diagonal and N is strictly upper-triangular and hence nilpotent.

9) Let T be a linear operator on F^2. Prove that if $v \neq 0$ is not a characteristic vector of T then it is a cyclic vector for T. Then prove that either T has a cyclic vector or else it is a scalar multiple of I.

9) Let T be the linear operator on $\mathbb{R}^3$ with matrix representation $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Show T has no cyclic vector.

9) For a linear operator T on a vectorspace V, prove that if T^2 has a cyclic vector then T has a cyclic vector. Is the converse true?

9) Let V be an n-dimensional vectorspace over F and let N be a nilpotent linear operator on V such that $N^{n-1} \neq 0$. Let v be any nonzero vector in V such that $N^{n-1} v \neq 0$. Show that v is a cyclic vector for N.

9) Let T be a linear operator on an n-dimensional vectorspace V. Suppose that T is diagonalizable.

(a) If T has a cyclic vector α show that T has n distinct characteristic values.

(b) If T has n distinct characteristic values, and if $B = \{b_1, \cdots, b_n\}$ is a basis of characteristic vectors for T, show that $b = b_1 + \cdots + b_n$ is a cyclic vector for T.

9) Let T be a linear operator on the dimension n vectorspace V. Suppose that T has a cyclic vector, v. Prove that if U is any linear operator which commutes with T then U is a polynomial in T.