1) Give an example of an infinite dimensional vector space V over $\mathbb{R}$, and linear operators T and S such that:

A) S is onto, but not one-to-one.

B) T is one-to-one, but not onto.

2) State true or false and justify: if V is a finite-dimensional vector space and W_1 and W_2 are subspaces of V such that $V = W_1 \oplus W_2$, then for any subspace W of V we have $W = (W \cap W_1) \oplus (W \cap W_2)$.

False. Example: $< (1,1) > \oplus < (0,1) >= \mathbb{R}$, but for $W = (1,0)$ we have $(< (1,0) > \cap < (1,1) >) \oplus (< (1,0) > \cap < (1,1) >)$.

3) Let F be a field, take $m, n \in \mathbb{Z}^+$ and let $A \in F^{m \times n}$ be an $m \times n$ matrix.

A) Define "row space of A"

B) Define "col space of A"

C) Prove that the dimension of the row space of A is equal to the dimension of the column space of A.

4) Let D be a principal ideal domain, let $n \in \mathbb{Z}^+$ and let $D^{(n)}$ denote a free D-module of rank n.

A) If L is a submodule of $D^{(n)}$, prove that L is a free D-module of rank $m \leq n$.

B) If L is a proper submodule of $D^{(n)}$, prove or disprove that the rank of L must be less than n.

5) Let D be a principal ideal domain and V and W denote free D-modules of rank 5 and 4 respectively. Assume that $\phi : V \to W$ is a D-module homomorphism, and that $B = \{v_1, \cdots, v_5\}$ is an ordered basis of V and $B' = \{w_1, \cdots, w_4\}$ is an ordered basis of W.

A) Define what is meant by the coordinate vector of $v \in V$ with respect to the basis B.

B) Describe how to obtain a matrix $A \in D^{4\times 5}$ so that left multiplication by A on D^5 represents $\phi : V \to W$ with respect to B and B'.

C) How does the matrix A change if we change the basis B by replacing v_2 by $v_2 + av_1$ for some $a \in D$?

D) How does the matrix A change if we change the basis B' by replacing w_2 by $w_2 + aw_1$ for some $a \in D$.

6) Let F be a subspace of the vector space $\mathbb{C}^{4\times 4}$ such that for every $A, B \in F$ we have $AB = BA$.

A) Demonstrate with an example that it is possible for there to exist in F five elements that are linearly independent over $\mathbb{C}$.

B) If there exists a matrix $A \in F$ having at least two distinct characteristic values, prove that $dimF \leq 4$.

7) Let V be a finite-dimensional vector space over an infinite field F and let $T : V \to V$ be a linear operator. Give to V the structure of a module over the polynomial rin $F[x]$ by defining $x \cdot \alpha = T(\alpha)$ for each $\alpha \in V$.

A) Outline a proof that V is a direct sum of cyclic $F[x]$-modules.

B) In terms of the expression for V as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that V have only finitely many T-invariant subspaces? Justify your answer.

8) Let M be a module over the integral domain D. Recall that a submodule N of M is said to be *pure* iff "whenever $y \in N$ and $a \in D$ are suc that there exists $x \in M$ with $ax = y$, then there exists $z \in N$ with $az = y$.

A) If N is a direct summand of M, prove that N is pure in M.

B) For $x \in M$, let $\bar{x} = x + N$ denote the coset representing the image of x in the quotient module M/N. If N is a pure submodule of M, and ann $\bar{x} = \{a \in D : a\bar{x} = 0\}$ is the principal ideal (d), prove that there exists $x' \in M$ such that $x + N = x' + N$and ann$x' = \{a \in D : ax' = 0\}$ is the principal ideal (d).

9) Let M be a finitely generated module over the polnoymial ring $F[x]$, where F is a field, and let N be a pure submodule of M. Prove that there exists a submodule L of M such that $N + L = M$ and $N \cap L = 0$.

10) Let V be a finite-dimensional complex inner product space and let $T : V \to V$ be a linear operator. Prove that T is self-adjoint iff $< T\alpha|\alpha >$ is real for every $\alpha \in V$.

11) Let V be an abelian group genereated by the elements a, b, c. Assume that $2a = 6b, 2b = 6c, 2c = 6a$, and that these three relations generate all the relations on a, b, c.

A) Write down a relation matrix for V.

B) Find generators x, y, z for V such that $V =< x > \oplus < y > \oplus < z >$ is the direct sum of cyclic subgroups generated by x, y, z.

C) Express your generators x, y, z in terms of a, b, c.

D) What is the order of V?

E) What is the order of the element a?

12) Let p be a prime integer and let $F = \mathbb{Z}/\shortmid\mathbb{Z}$ be the field with p elements. Let V be a vectorspace over F and $T : V \to V$ a linear operator. Assume that T has characteristic polynomial x^3 and minimal polynomial x^2.

A) Express V as a direct sum of cyclic $F[x]$-modules.

B) How many 1-dimensional T-invariant subspaces does V have?

C) How many of the 1-dimsenional T-invariant subspaces of V are direct sumands of V?

D) How many 2-dimsenional T-invariant subspaces does V have?

E) How many 2-dimensional T-invariant subspaces of V are direct sumands of V?

13) Let V be a 5-dimensional vectorspace over the field F and let $T : V \to V$ be a linear operator such that rank $T = 1$. List all polynomials $p(x) \in F[x]$ that are possibly the minimal polynomial of T. Explain.

14) Let F be a field.

A) What is the dimension of the vectorspace of all 3-linear functions $D : F^{3\times 3} \to F$?

B) What is the dimesnions of the vectorspace of all 3-linear alternating functions $D : F^{3\times 3} \to F$.

15) Let V be a finite-dimensional vectorspace over a field F. Prove that a linear operator $T : V \to V$ has a cyclic vector iff every linear operator $S : V \to V$ that commutes with T is a polynomial in T.