A NOTE ON THE FACTORIZATION THEOREM OF TORIC BIRATIONAL MAPS AFTER MORELLI AND ITS TOROIDAL EXTENSION

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This note is a result of series of seminars held by the authors during the summer of 1997 and continued from then on, toward a thorough understanding of the factorization theorem of toric birational maps (morphisms) into equivariant blowups and blowdowns by [Morelli1] (cf.[Wlodarczyk]). Our purpose is two-fold. The first is to give a coherent presentation of the proof in [Morelli1], modifying some (mostly minor except for a couple of points in the process of π-desingularization and in showing that weak factorization implies strong factorization) discrepancies found by H. King [King2] and by the authors in the due course of the seminars checking the original arguments. (As of Jan. 1998 we learned from Prof. Fulton that Morelli himself offers correction in his homepage [Morelli2] to the discrepancies in the π-desingularization process found by H. King. We need some clarification, as is presented in this note, to understand the correction. We thank Prof. Morelli for guiding us toward a better understanding through private communication.) It is a mere attempt to see in a transparent way the beautiful and brilliant original ideas of [Morelli1,2] by sweeping dust off the surface. The second is the generalization to the toroidal case, whose details are worked out as a part of Ph. D. thesis of the third author.

Though it may be said that the toroidal generalization is straightforward and even implicit in the original papers [Morelli1,2] (cf.[Wlodarczyk]), we would like to emphasize its importance in a more far-reaching problem formulated as below, with a view toward its application to the factorization problem of general birational maps.

In a most naive way the "far-reaching" problem can be stated as follows: Let $f : X \to Y$ be a morphism (one may put the condition "with connected fibers" if he wishes) between nonsingular projective varieties. By replacing X and Y with their

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modifications X' and Y', how "NICE" can one make the morphism $f' : X' \to Y'$?

$$\begin{tikzcd} X & X' \arrow[l] \arrow[d, "f'"] \\ Y \arrow[from=u, "f"] & Y' \arrow[l] \end{tikzcd}$$

Depending upon how we interpret the word "NICE" mathematically and what restrictions we put on the modifications, we get the corresponding interesting questions such as semi-stable reduction (when the morphism f' is "NICE" iff every fiber is reduced with only simple normal crossings with $\dim Y = 1$ and the modifications for Y are restricted to finite morphisms while the modifications for X are restricted to smooth blowups after base change), resolution of hypersurface singularities (when the morphism f' is "NICE" iff every fiber has only simple normal crossings, this time not necessarily reduced, with $\dim Y = 1$, and no modification for Y and only smooth blowups are allowed for X). When f is birational and we require f' to be an isomorphism in order for it to be "NICE", restricting the modifications of X and Y to be smooth blowups, we obtain the long standing (and perhaps notorious) factorization problem for general birational morphisms.

Our interpretation is that we put "toroidal" for the word "NICE" and restrict the modifications of X and Y to be only blowups with smooth centers.

Toroidalization Conjecture. *Let $f : X \to Y$ be a morphism between nonsingular projective varieties. Then there exist sequences of blowups with smooth centers for X and Y so that the induced morphism $f' : X' \to Y'$ is toroidal:*

$$\begin{tikzcd} X \arrow[d, "f"] & X' \arrow[l, "\text{smooth blowups}"'] \arrow[d, "f'"] & \textit{toroidal} \\ Y & Y' \arrow[l, "\text{smooth blowups}"'] & \end{tikzcd}$$

The conjecture is closely related to the recent work of [Abramovich-Karu], which introduces the notion of "toroidal" morphisms explicitly for the first time, though implicitly it can be recognized in [KKMS]. By only requiring "NICE" morphisms to be toroidal instead of being isomorphisms, we can start dealing not only with birational morphisms but also with fibering morphisms between varieties of different dimensions. This seems to give us more freedom to seek some inductional structure. Actually we expect that the powerful inductive method of [Bierstone-Milman] for the canonical resolution of singularities, proceeding from the hypersurface case with only one defining equation to the general case with several defining equations through the ingeneous use of invariants, should be modified to be applied to our toroidalization problem, proceeding similarly from the case $\dim Y = 1$ to the general case $\dim Y > 1$.

This interpretation not only generalizes the statement of the classical factorization problem but also gives the following approach to it:

A Conjectural Approach to the Factorization Problem via Toroidalization. *Given a birational morphism $f : X \to Y$,*

(I) Make it "toroidal" $f' : X' \to Y'$ modifying X and Y into X' and Y' by blowing up with smooth centers via some Bierstone-Milman type argument,

and then

(II) Factorize the toroidal birational morphism $f' : X' \to Y'$ into (equivariant) smooth blowups and blowdowns by applying the toroidal version of the method of [Morelli1,2] (or [Wlodarczyk]).

The toroidalization conjecture and the (strong) factorization of toroidal birational morphisms would imply the (strong) factorization of general birational maps between nonsingular projective varieties.

This line of ideas came up in our conversation as a day-dreaming inspired by [deJong], only to find out later that an almost identical approach was already presented in [King1] and has been pursued by him in reality, who has (privately) announced the affirmative solution to the toroidalization conjecture in the case $\dim X = 3$. Actually our formulation above follows his presentation in [King1]. He has also read [Morelli1] carefully and his correspondence with Morelli himself was kindly communicated to us by E. Bierstone. We thank both professors for their generosity sharing their ideas with us and our indebtedness to them is both explicitly and implicitly clear as well as to the original papers [Morelli1,2] and [Wlodarczyk]. Another big inspiration for the factorization problem comes from the recent result of [Cutkosky1], which affirmatively solves the local factorization problem in dimension 3 using the valuation theory. We thank Prof. Cutkosky for kindly teaching us his ideas of the valuation theory method via preprints and private conversations. In response, we communicated to him our idea above for the global factorization, which turns out to be very similar to the idea of [Christensen] in the local factorization:

(I) First "monomialize" the given local birational morphism via the valuation theory (The monomialization is slightly stronger than the toroidalization in the local case, though in the global case one can only hope to toroidalize a map but not monomialize.), then

(II) Factorize the local monomial birational morphism.

[Cutkosky2,3] achieves the local factorization in arbitrary dimension along this line of ideas, extending his method of valuation theory.

(After monomializing the map in (I), which is the most sutble and difficult part, [Cutkosky2] refers to the results of Morelli in (II). [Cutkosky3] factorizes the monomialized map in his own algorithm in (II) avoiding the use of results of Morelli, which contained some discrepancies at the time. Though his algorithm is of interset in its own sake, it only produces the weak factorization. If we apply the strong factorization theorem of this note by Morelli in (II), then [Cutkosky2,3] yields the strong factorization theorem in the local case.)

Our organization, as being a note to [Morelli1,2], follows exactly the structure of the original paper [Morelli1,2] with one last section on the toroidal case added.

We remark that [Reid1] gives the factorization of toric birational maps into extremal divisorial contractions and flips by establsishing the Minimal Model Program for toric varieties in the realm of the Mori theory. We also remark that recently a new algorithm called the Sarkisov Program has emerged (cf.[Sarkisov][Reid2]) to factorize birational maps among uniruled varieties. Though it is only established in dimension 3 in general (cf.[Corti]), the toric case is rather straightforward in arbitrary dimension (cf.[Matsuki]).

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§1. Introduction

The purpose of this section is to present the basic ideas of the brilliant solution of [Morelli1,2] (See also [Wlodarczyk].) to the following conjecture of Oda [Miyake-Oda]:

Oda's Conjecture. *Every proper and equivariant birational map $f : X_\Delta \dashrightarrow X_{\Delta'}$ ("proper" in the sense of [Iitaka]) between two nonsingular toric varieties can be factorized into a sequence of blowups and blowdowns with centers being smooth closed orbits.*

If we allow the sequence to consist of blowups and blowdowns in any order, then the factorization is called **weak**.

If we insist on the sequence to consist of blowups immediately followed by blowdowns, then the factorization is called **strong**.

As toric varieties X_Δ correspond to the fans Δ in $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$, where N is the lattice of 1-parameter subgroups of the torus, and blowups to the smooth star subdivisions of Δ, we can reformulate Oda's conjecture in the following purely combinatorial language:

Oda's conjecture in terms of fans. *Let Δ and Δ' be two nonsingular fans in $N_{\mathbb{Q}}$ with the same support. Then there is a sequence of smooth star subdivisions and inverse operations called smooth star assemblings starting from Δ and ending with Δ'.*

If we allow the sequence to consist of smooth star subdivisions and smooth star assemblings in any order, then the factoization is called weak.

If we insist on the sequence to consist of smooth star subdivisions immediately followed by smooth star assemblings, then the factorization is called strong.

In order to understand Morelli's strategy toward the solution of Oda's conjecture, we look at the following simple example.

We take Δ and Δ' to consist of the maximal cones in $N \cong \mathbb{Z}^3 \subset N_{\mathbb{Q}} = N \otimes \mathbb{Q}$

$$\begin{aligned} \Delta &= \{\gamma_{123}, \gamma_{124} \text{ and their proper faces}\} \\ \Delta' &= \{\gamma_{134}, \gamma_{234} \text{ and their proper faces}\} \end{aligned}$$

where $\gamma_{ijk} =< v_i, v_j, v_k >$

with $v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1), v_4 = (1,1,-1)$.

Then we observe that by taking the star subdivisin $\tilde{\Delta}$ by the vector

$$v_1 + v_2 = v_3 + v_4$$

the transformation from Δ to Δ' can be factorized into a smooth star subdivision immediately followed by a smooth star assembling

$$\Delta \leftarrow \tilde{\Delta} \rightarrow \Delta'$$

as asserted by Oda's conjecture.

Morelli's great idea is to incorporate all the information of this factorization as a "cobordism" Σ in the space $N_{\mathbb{Q}}^+ = N_{\mathbb{Q}} \oplus \mathbb{Q}$, adding one more vertical direction to the original space $N_{\mathbb{Q}}$. The lower face $\partial_-\Sigma$ and the upper face $\partial_+\Sigma$ of Σ are isomorphic to Δ and Δ', respectively. Namely, we take

$$\begin{aligned} &\Sigma = \{\sigma\} \subset N_{\mathbb{Q}}^+ = N_{\mathbb{Q}} \oplus \mathbb{Q} \\ &\text{where } \sigma =< \rho_1, \rho_2, \rho_3, \rho_4 > \\ &\text{with } \rho_1 = (v_1, 0), \rho_2 = (v_2, 0), \rho_3 = (v_3, 1), \rho_4 = (v_4, 1) \end{aligned}$$

with the projection

$$\pi : N_{\mathbb{Q}}^+ = N_{\mathbb{Q}} \oplus \mathbb{Q} \to N_{\mathbb{Q}}.$$

The lower face $\partial_-\Sigma$ of Σ

$$\partial_-\Sigma = \{< \rho_1, \rho_2, \rho_3 >, < \rho_1, \rho_2, \rho_4 > \text{ and their proper faces}\}$$

maps isomorphically onto Δ by the projection π and so does the upper face $\partial_+\Sigma$

$$\partial_+\Sigma = \{< \rho_1, \rho_3, \rho_4 >, < \rho_2, \rho_3, \rho_4 > \text{ and their proper faces}\}$$

isomorphically onto Δ'.

Moreover, since σ does not map isomorphically onto its image by π i.e., since σ is π-dependent, we have the linear relation among the the primitive vectors v_i of the projections of the generators ρ_i of σ

$$v_1 + v_2 - v_3 - v_4 = 0.$$

From this linear relation, we can read off the point

$$v_1 + v_2 = v_3 + v_4$$

by which we have to subdivide Δ and Δ' to reach the common refinement $\tilde{\Delta}$.

In short, we can realize the factorization from consructing the cobordism.

Morelli's Idea for Factorization. *Let Δ and Δ' be two nonsingular fans in $N_{\mathbb{Q}}$ with the same support. Then we can realize the (weak) factorization by constructing a cobordism Σ, a simplicial fan in $N_{\mathbb{Q}}^+ = N_{\mathbb{Q}} \oplus \mathbb{Q}$ such that*

(o) the lower face $\partial_-\Sigma$ and the upper face $\partial_+\Sigma$ of Σ map isomorphically onto Δ and Δ' by the projection π

$$\begin{aligned} \pi &: \partial_-\Sigma \xrightarrow{\sim} \Delta \\ \pi &: \partial_+\Sigma \xrightarrow{\sim} \Delta', \end{aligned}$$

(i) Σ is π-nonsingular (See §3 for the precise definition.),

(ii) Σ is collapsible (See §4 for the precise definition.).

In fact, let σ be a minimal simplex in Σ which is π-dependent. (We call such simplex σ a circuit.) If σ is generated by the extremal rays ρ_i

$$\sigma = < \rho_1, \rho_2, \cdots, \rho_k >,$$

then we have the linear relation among the primitive vectors v_i of the projections of the generators ρ_i

$$\Sigma_{i=1}^k r_i v_i = 0.$$

Now by the π-nonsingularity of Σ and minimality of σ, it follows that we may assume that all the coefficients are either $+1$ or -1 after rescaling. Thus after renumbering the v_i, we may assume that the linear relation is given by

$$v_1 + v_2 + \cdots + v_l - v_{l+1} - \cdots - v_k = 0.$$

We then observe that the star subdivision by the point

$$v_1 + v_2 + \cdots + v_l = v_{l+1} + \cdots + v_k$$

gives the factorization between the lower face $\partial_-\sigma$ and the upper face $\partial_+\sigma$. Or more generally, we obtain the factorization between the lower face $\partial_-\overline{Star(\sigma)}$ and upper face $\partial_+\overline{Star(\sigma)}$ of the closed star $\overline{Star(\sigma)}$ of σ, where

$$\overline{Star(\sigma)} = \{\zeta \in \Sigma; \zeta \subset \eta \supset \sigma \text{ for some } \eta \in \Sigma\}.$$

The π-nonsingularity also guarantees that all the lower and upper faces and the common refinement obtained through the star subdivision are nonsingular and the star subdivision is smooth.

This achieves the (weak) factorization for $\overline{Star(\sigma)}$ for one circuit σ of Σ. In order to achieve the (weak) factorization for the entire $\Sigma = \cup\overline{Star(\sigma)} \cup \partial_+\Sigma$, where the union is taken over all the circuits σ in Σ, we have to coordinate the way we take the (weak) factorizaton of the circuits altogether. This is done by requiring the collapsibility of the cobordism Σ.

In §2, we construct a cobordism Σ between two simplicial fans Δ and Δ'. The simplicial cobordism constructed in this section only satisfies the condition (o) above of Morelli's idea. The construction is done via the slick use of Sumihiro's equivariant completion theorem.

In §3, we discuss the (weak) factorization between the lower face $\partial_-\overline{Star(\sigma)}$ and upper face $\partial_+\overline{Star(\sigma)}$, which we call the bistellar operation, more in detail assuming the π-nonsingularity.

In §4, we achieve the condition (ii), the collapsibility for the simplicial cobordism Σ. By star subdividing further Σ, obtained in §2, we attain a cobordism $\tilde{\Sigma}$ which is projective via the use of toric version of Moishezon's theorem. Projectivity implies collapsibility, achieving a collapsible and simplicial cobordism between $\partial_-\tilde{\Sigma}$ and

$\partial_+\tilde{\Sigma}$. We can explicitly construct a collapsible and simplicial cobordism Σ_1 (resp. Σ_2) between Δ and $\partial_-\tilde{\Sigma}$ (resp. between $\partial_+\tilde{\Sigma}$ and Δ'), as the latter is obtained through star subdivisions (resp. star assemblings) from the former. Now we only have to take the composition $\Sigma_1 \circ \tilde{\Sigma} \circ \Sigma_2$ to be the new collapsible and simplicial cobordism Σ between Δ and Δ'.

§5 is the most sutble and difficult part of the proof, achieving the π-nonsingularity of the cobordism Σ. We introduce the invariant "π-multiplicity profile" of a simplicial cobordism, which measures how far Σ is from being π-nonsingular, and observe that it strictly drops after some appropriate star subdivisions. By the descending chain condition on the set of the π-multiplicity profiles, we reach the π-nonsingularity after finitely many star subdivisions.

§§2 $\sim$ 5 provide the weak factorization, solving the weak form of Oda's conjecture affirmatively. The results are summarized in §6.

We should emphasize that the weak form of Oda's conjecture is also solved by [Wlodarczyk] along a similar line of ideas but in a more combinatorial language.

In §7, we finally show the strong factorization, based upon the weak factorization of the previous sections. We obtain $\tilde{\Sigma}$ by further star subdividing the cobordism Σ corresponding to the weak factorization between Δ and Δ', without affecting the lower face of Σ but possibly smooth star subdividing the upper face of Σ, so that the bistellar operations of the circuits in $\tilde{\Sigma}$ only provide blowups starting from the lower face. In other words, we manipulate so that all the circuits in $\tilde{\Sigma}$ are "pointing up". We achieve the strong factorization

$$\Delta \cong \partial_-\Sigma = \partial_-\tilde{\Sigma} \leftarrow \partial_+\tilde{\Sigma} \rightarrow \partial_+\Sigma \cong \Delta',$$

the first left arrow representing a sequence of blowups and the second right arrow representing a sequence of blowdowns immediately after.

§8 discusses the generalization to the toroidal case. All the arguments above for the toric case can be lifted immediately to the toroidal case, except for the existence of a cobordism and π-collapsibility, where we used the global results like Sumihiro's and Moishezon's theorems only valid in the toric case. We circumvent these difficulties by a simple trick embedding a toroidal conical complex into a usual toric fan after barycentric subdivisions.

§2. Cobordism

We follow the usual notation and terminology concerning the toric varieties X_Δ and their corresponding fans Δ, as presented in [Danilov][Fulton] or [Oda].

We recall the notion of star subdivisions of a fan Δ, the key operation repeatedly used in this note.

Definition 2.1. *Let $\tau \in \Delta$ be a cone in a fan Δ. Let ρ be a ray passing in the relative interior of τ. (Note that such $\tau \in \Delta$ containing ρ in its relative interior is uniquely determined once ρ is fixed.) Then we define the star subdivision $\rho \cdot \Delta$ of Δ with respect to ρ to be*

$$\rho \cdot \Delta = (\Delta - Star(\tau)) \cup \{\rho + \tau' + \nu; \tau' \text{ a proper face of } \tau, \nu \in link_\Delta(\tau)\}$$

where

$$\begin{aligned} Star(\tau) &= \{\zeta \in \Delta; \zeta \supset \tau\} \\ \overline{Star(\tau)} &= \{\zeta \in \Delta; \zeta \subset \eta \text{ for some } \eta \in Star(\tau)\} \\ link_\Delta(\tau) &= \{\zeta \in \overline{Star(\tau)}; \zeta \cap \tau = \emptyset\} \end{aligned}$$

We call the inverse of a star subdivision a star assembling.

When τ is generated by extremal rays ρ_i

$$\tau = < \rho_1, \cdots, \rho_l >$$

with the primitive vectors

$$v_i = n(\rho_i)$$

and the ray ρ is generated by the vector

$$v_1 + \cdots + v_l,$$

the star subdivision is called the barycentric subdivision with respect to τ.

When Δ is nonsingular, the barycentric subdivision with respect to a face τ is called a smooth star subdivision and its inverse a smooth star assembling.

The notion of a cobordism as defined below sits in the center of Morelli's idea.

Definition 2.2. *Let Δ and Δ' be two fans in $N_\mathbb{Q} = N \otimes \mathbb{Q}$ with the same support, where N is the lattice of 1-parameter subgroups of the torus. A cobordism Σ is a fan in $N_\mathbb{Q}^+ = (N \oplus \mathbb{Z}) \otimes \mathbb{Q} = N_\mathbb{Q} \oplus \mathbb{Q}$ equipped with the map induced by the natural projection*

$$\pi : N_\mathbb{Q}^+ = N_\mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q}$$

such that

(o) π gives an isomorphism between $\partial_-\Sigma$ and Δ (resp. $\partial_+\Sigma$ and Δ') as linear complexes, i.e., there is a 1-to-1 correspondence between the cones σ_- of $\partial_-\Sigma$ (resp. σ'_+ of $\partial_+\Sigma$) and the cones σ of Δ (resp. σ' of Δ') such that $\pi : \sigma_- \to \sigma$ (resp.

$\pi : \sigma'_+ \to \sigma'$) is an linear isomorphism for each σ_- (resp. σ'_+) and its corresponding σ (resp. σ')

$$\pi : \partial_-\Sigma \overset{\sim}{\to} \Delta$$
$$(resp. \pi : \partial_+\Sigma \overset{\sim}{\to} \Delta')$$

where

$$\begin{aligned}\partial_-\Sigma =&\{\tau \in \Sigma; (x, y-\epsilon) \notin Supp(\Sigma)\\ &\text{for any } (x,y) \in \tau \text{ with } x \in N_{\mathbb{Q}}, y \in \mathbb{Q} \text{ and any } \epsilon > 0\}\\ (resp. \partial_+\Sigma =&\{\tau \in \Sigma; (x, y+\epsilon) \notin Supp(\Sigma)\\ &\text{for any } (x,y) \in \tau \text{ with } x \in N_{\mathbb{Q}}, y \in \mathbb{Q} \text{ and any } \epsilon > 0\})\end{aligned}$$

(i) the support of Σ lies between the lower face $\partial_-\Sigma$ and the upper face $\partial_+\Sigma$

$$\begin{aligned}Supp(\Sigma) = \{&(x,y) \in N^+_{\mathbb{Q}}; x \in Supp(\Delta) = Supp(\Delta') \text{ and } y^x_- \leq y \leq y^x_+\\ &\text{where } (x, y^x_-) \in Supp(\partial_-\Sigma) \text{ and } (x, y^x_+) \in Supp(\partial_+\Sigma).\}\end{aligned}$$

Theorem 2.3. *Let Δ and Δ' be two fans in $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$ with the same support. Then there exists a cobordism Σ between Δ and Δ'. If Δ and Δ' are both simplicial, we may also require Σ to be simplicial.*

Proof.

First we embed Δ into $N^+_{\mathbb{Q}}$ so that the projection π maps isomorphically back onto Δ. Namely, we take the fan Δ_- in $N^+_{\mathbb{Q}}$ consisting of the cones σ_- of the form

$$\sigma_- =< (v_1, -1), \cdots, (v_k, -1) >$$

where the corresponding cone $\sigma \in \Delta$ is generated by the extremal rays ρ_i

$$\sigma =< \rho_1, \cdots, \rho_k >$$

with the primitive vectors

$$v_i = n(\rho_i).$$

Similarly we embed Δ' into $N^+_{\mathbb{Q}}$ so that the projection π maps isomorphically back onto Δ'. Namely, we take the fan Δ'_+ in $N^+_{\mathbb{Q}}$ consisting of the cones σ'_+ of the form

$$\sigma'_+ =< (v'_1, +1), \cdots, (v'_k, +1) >$$

where $\sigma' \in \Delta'$ is generated by the extremal rays ρ'_i

$$\sigma' =< \rho'_1, \cdots, \rho'_k >$$

with the primitive vectors

$$v'_i = n(\rho'_i).$$

We take Σ° to be the fan in $N_{\mathbb{Q}}^+$ consisting of the cones in Δ_- and ${\Delta'}_+$ and the cones ζ of the form

$$\zeta = < (v, -1), (v, +1) >$$

where the v vary among all the primitive vectors for the extremal rays ρ_v such that ρ_v is a generator for some $\sigma \in \Delta$ and some $\sigma' \in \Delta'$ simultaneously.

Now by Sumihiro's equivariant completion theorem [Sumihiro], there exists a fan Γ with $\mathrm{Supp}(\Gamma) = N_{\mathbb{Q}}^+$ and containing Σ° as a subfan.

We only have to take Σ to be

$$\Sigma = \{\tau \in \Gamma; \mathrm{Supp}(\tau) \subset S\}$$

where the set S is described as below

$$\begin{aligned} S = \{(x, y) \in N_{\mathbb{Q}}^+; x \in \mathrm{Supp}(\Delta) = \mathrm{Supp}(\Delta'), y_-^x \leq y \leq y_+^x \\ \text{with } (x, y_-^x) \in \Delta_-, (x, y_+^x) \in {\Delta'}_+\}. \end{aligned}$$

If Δ and Δ' are both simplicial, then we construct a cobordism Σ as above first, which may not be simplicial. We take all the cones in Σ which are not simplicial, and give them the partial order according to the inclusion relation. We take a succession of barycentric subdivisions with respect to these cones in the order compatible with the partial order, starting with the maximal ones. The resulting fan $\tilde{\Sigma}$ is simplicial with the property

$$\begin{aligned} \pi : \partial_- \tilde{\Sigma} = \partial_- \Sigma = \Delta_- \overset{\sim}{\to} \Delta \\ \pi : \partial_+ \tilde{\Sigma} = \partial_+ \Sigma = \Delta'_+ \overset{\sim}{\to} \Delta', \end{aligned}$$

providing a simplicial cobordism between Δ and Δ'.

§3. Circuits and Bistellar Operations

In this section, we discuss how to read off the information on the factorization from the circuits of a π-nonsingular cobordism.

Definition 3.1. *Let Σ be a simplicial fan in $N_{\mathbb{Q}}^+ = N_{\mathbb{Q}} \oplus \mathbb{Q} = (N \oplus \mathbb{Z}) \otimes \mathbb{Q}$ with the natural projection $\pi : N_{\mathbb{Q}}^+ \to N_{\mathbb{Q}}$.*

A cone $\sigma \in \Sigma$ is π-independent iff $\pi : \sigma \to \pi(\sigma)$ is an isomorphism. Otherwise σ is π-dependent.

A cone $\sigma \in \Sigma$ is called a circuit iff it is minimal and π-dependent, i.e., iff σ is π-dependent and any proper subface of σ is π-independent.

A cone $\sigma \in \Sigma$ is π-nonsingular iff σ is π-independent and $\pi(\sigma)$ is nonsingular as a cone in $N_{\mathbb{Q}}$ with respect to the lattice N. We say the fan Σ is π-nonsingular iff all the π-independent cones in Σ are π-nonsingular.

The following theorem describes the transformation from the lower face $\partial_-\sigma$ to the upper face $\partial_+\sigma$ (or more precisely the transformation from $\pi(\partial_-\sigma)$ to $\pi(\partial_+\sigma)$), which we call the bistellar operation, of a circuit σ of a π-nonsingular simplicial fan Σ. (More generally the theorem describes the transformation from the lower face $\partial_-\overline{Star(\sigma)}$ to the upper face $\partial_+\overline{Star(\sigma)}$ of the closed star of a circuit σ.) It corresponds to a smooth blowup immediately followed by a smooth blowdown.

Theorem 3.2. *Let Σ be a simplicial and π-nonsingular fan in $N_{\mathbb{Q}}^+$. Let $\sigma \in \Sigma$ a circuit generated by the extremal rays ρ_i*

$$\sigma = < \rho_1, \cdots, \rho_k > .$$

From each extremal ray, we take the vector of the form $(v_i, w_i) \in N_{\mathbb{Q}}^+ = N_{\mathbb{Q}} \oplus \mathbb{Q}$ where

$$v_i = n(\pi(\rho_i))$$

is the primitive vector of the projection.

(i) There is the unique linear relation among the v_i (up to renumbering and rescaling) of the form

$$\Sigma r_\alpha v_\alpha = v_1 + \cdots + v_l - v_{l+1} - \cdots - v_k = 0 \text{ for some } 0 \leq l \leq k$$

with

$$\Sigma r_\alpha w_\alpha = w_1 + \cdots + w_l - w_{l+1} - \cdots - w_k > 0.$$

(ii) All the maximal faces γ_i (resp. γ_j) of $\partial_-\sigma$ (resp. $\partial_+\sigma$) are of the form

$$\gamma_i = < \rho_1, \cdots, \overset{\vee}{\rho_i}, \cdots, \rho_l, \rho_{l+1}, \cdots, \rho_k > \quad 1 \leq i \leq l$$
$$(\text{resp.}\gamma_j = < \rho_1, \cdots, \rho_l, \rho_{l+1}, \cdots, \overset{\vee}{\rho_j}, \cdots, \rho_k > \quad l+1 \leq j \leq k$$

(iii) Let l_σ be the extremal ray generated by the vector

$$v_1 + \cdots + v_l = v_{l+1} + \cdots + v_k.$$

The smooth star subdivision of $\pi(\partial_-\sigma)$ with respect to l_σ coincides with the smooth star subdivision of $\pi(\partial_+\sigma)$ with respect to l_σ, whose maximal faces $< \pi(\gamma_{ij}), l_\sigma >$ are of the form

$$< \pi(\gamma_{ij}), l_\sigma > = < \pi(\rho_1), \cdots, \pi(\overset{\vee}{\rho_i}), \cdots, \pi(\rho_l), \pi(\rho_{l+1}), \cdots, \pi(\overset{\vee}{\rho_j}), \cdots, \pi(\rho_k), l_\sigma > .$$

Thus the transformation from $\pi(\partial_-\sigma)$ to $\pi(\partial_+\sigma)$ is a smooth star subdivision followed immediately after by a smooth star assembling. We call the transformation the bistellar operation.

Similarly, the transformation from the lower face $\partial_-\overline{Star(\sigma)}$ to the upper face $\partial_+\overline{Star(\sigma)}$ is a smooth star subdivision followed immediately after by a smooth star assembling.

Proof.

(i) Since σ is a circuit, it is π-dependent and minimal by definition. Hence we have a linear relation

$$\Sigma r_i v_i = 0 \quad r_i \neq 0 \quad \forall i.$$

Since σ is simplicial, the ρ_i are linearly independent in $N_{\mathbb{Q}}^+$ and hence

$$\Sigma r_i w_i \neq 0.$$

We choose the signatures of the r_i so that

$$\Sigma r_i w_i > 0.$$

We only have to prove

$$|r_1| = |r_2| = \cdots = |r_k|.$$

In fact, since σ is π-nonsingular, we have

$$\begin{aligned} 1 &= |\det(\overset{\vee}{v_1}, v_2, \cdots, v_k)| \\ &= |\det(v_1, \cdots, \overset{\vee}{v_i}, \cdots, v_k)| \\ &= |\det(\frac{1}{r_1}(\Sigma_{\alpha\neq 1} r_\alpha v_\alpha), v_2, \cdots, \overset{\vee}{v_i}, \cdots, v_k)| \\ &= |\frac{r_i}{r_1}\det(\overset{\vee}{v_1}, v_2, \cdots, v_k)|, \end{aligned}$$

which implies

$$|r_1| = |r_i| \quad \forall i.$$

(ii) Note that since σ is a circuit, any maximal face γ of σ belongs either to $\partial_-\sigma$ or to $\partial_+\sigma$, exclusively.

Suppose

$$\gamma_i = < \rho_1, \cdots, \overset{\vee}{\rho_i}, \cdots, \rho_k > \in \partial_-\sigma.$$

Then since σ is a circuit, for any point

$$p = \Sigma_{\alpha\neq i} c_\alpha \rho_\alpha \in RelInt(\gamma_i) \text{ with } c_\alpha > 0,$$

we have

$$p + (0, \epsilon) \in \sigma \text{ for } 0 < \epsilon << 0.$$

By setting

$$\epsilon = t_\epsilon \cdot \Sigma r_\alpha w_\alpha \text{ for } t_\epsilon > 0,$$

we obtain

$$p + (0, \epsilon) = \Sigma_{\alpha \neq i}(c_\alpha + t_\epsilon r_\alpha)(v_\alpha, w_\alpha) + t_\epsilon r_i(v_i, w_i) \in \sigma,$$

which implies

$$\begin{aligned} &c_\alpha + t_\epsilon r_\alpha > 0 \text{ for } \alpha \neq i \\ &t_\epsilon r_i > 0. \end{aligned}$$

Therefore, we have

$$r_i > 0.$$

Similarly, if

$$\gamma_j =< \rho_1, \cdots, \overset{\vee}{\rho_j}, \cdots, \rho_k >\in \partial_+ \sigma,$$

then we have

$$r_j < 0.$$

This proves the assertion (ii).

The assertion (iii) follows immediately from (i) and (ii).

The assertion about the transformation from $\partial_- \overline{Star(\sigma)}$ to $\partial_+ \overline{Star(\sigma)}$ is an easy consequence of the description of the transformation from $\partial_- \sigma$ to $\partial_+ \sigma$.

This completes the proof of Theorem 3.2.

§4. Collapsibility

Let Σ be a simplicial cobordism between simplicial fans Δ and Δ'. Noting that

$$\Sigma = \cup_\sigma \overline{Star(\sigma)} \cup \partial_+\Sigma$$

where the union is taken over the circuits σ, we may try to factorize the transformation from $\Delta \cong \partial_-\Sigma$ to $\Delta' \cong \partial_+\Sigma$ into smooth star subdivisions and smooth star assemblings by replacing $\partial_-\overline{Star(\sigma)}$ with $\partial_+\overline{Star(\sigma)}$, if Σ is π-nonsingular, based upon the analysis of §3. If we think of the cobordism built up out of bubbles $\overline{Star(\sigma)}$, this process might be considered as a succession of "collapsing" these bubbles. The following simple example shows that this succession of collapsing, which should correspond to the factorization into smooth star subdivisions and smooth star assembling, is not always possible, unless we can arrange the way we break these bubbles in a certain order. This possiblility for the certain nice arrangement is what we call "collapsibility" in this section.

Example 4.1. *We take two sets of vectors in $N_{\mathbb{Q}} = \mathbb{Z}^2 \otimes \mathbb{Q}$*

$$\{v_1 = (1,0), v_2 = (0,1), v_3 = (-1,0), v_4 = (0,-1)\}$$
$$\{v'_1 = (1,1), v'_2 = (-1,1), v'_3 = (-1,-1), v'_4 = (1,-1)\}$$

and fans Δ and Δ' whose maximal cones consist of

$$\Delta \ni \sigma_{12} =< v_1, v_2 >, \sigma_{23} =< v_2, v_3 >, \sigma_{34} =< v_3, v_4 >, \sigma_{41} =< v_4, v_1 >$$
$$\Delta' \ni \sigma'_{12} =< v'_1, v'_2 >, \sigma'_{23} =< v'_2, v'_3 >, \sigma'_{34} =< v'_3, v'_4 >, \sigma'_{41} =< v'_4, v'_1 >$$

If we take the simplicial fan Σ in $N_{\mathbb{Q}}^+$ to consist of the maximal cones

$$\begin{aligned}
\sigma_{124'} &=< (v_1,0),(v_2,0),(v'_4,1) > \\
\sigma_{231'} &=< (v_2,0),(v_3,0),(v'_1,1) > \\
\sigma_{342'} &=< (v_3,0),(v_4,0),(v'_2,1) > \\
\sigma_{413'} &=< (v_4,0),(v_1,0),(v'_3,1) > \\
\sigma_{4'1'2} &=< (v'_4,1),(v'_1,1),(v_2,0) > \\
\sigma_{1'2'3} &=< (v'_1,1),(v'_2,1),(v_3,0) > \\
\sigma_{2'3'4} &=< (v'_2,1),(v'_3,1),(v_4,0) > \\
\sigma_{3'4'1} &=< (v'_3,1),(v'_4,1),(v_1,0) >,
\end{aligned}$$

then Σ is a simplicial π-nonsingular cobordism between Δ and Δ'.

Observe, however, that we cannot "collapse" any one of the maximal cones $\sigma_{ijk'}$ or $\sigma_{i'j'k}$ to replace $\partial_-\sigma_{ijk'}$ with $\partial_+\sigma_{ijk'}$ or to replace $\partial_-\sigma_{i'j'k}$ with $\partial_+\sigma_{i'j'k}$. We cannot see the entire $\partial_-\sigma_{ijk'}$ or $\partial_-i'j'k$ from below or the entire $\partial_+\sigma_{ijk'}$ or $\partial_+\sigma_{i'j'k}$ from above. In fact, the circuit graph attached to Σ as defined below is a directed cycle consisting of 8 vertices

$$\begin{aligned}
&\sigma_{124'} \to \sigma_{4'1'2} \to \sigma_{231'} \to \sigma_{1'2'3} \\
\to &\sigma_{342'} \to \sigma_{2'3'4} \to \sigma_{413'} \to \sigma_{3'4'1} \to \sigma_{124'}.
\end{aligned}$$

Definition 4.2. *Let Σ be a simplicial fan in $N_{\mathbb{Q}}^{+}$. We define a directed graph, which we call the circuit graph of Σ, as follows: The vertices of the circuit graph of Σ consist of the circuits σ of Σ. We draw an edge from a circuit σ to another σ' iff there is a point $p \in \partial_{+}\overline{Star(\sigma)} \cap \partial_{-}\overline{Star(\sigma')}$ such that*

$$p - (0,\epsilon) \in \overline{Star(\sigma)}, p + (0,\epsilon) \in \overline{Star(\sigma')} \text{ for } 0 << \epsilon << 1.$$

We say Σ is collapsible if the circuit graph contains no directed cycle. When Σ is collapsible, the circuit graph determines a partial order among the circuits: $\sigma \leq \sigma'$ iff there is an edge $\sigma \to \sigma'$.

Theorem 4.3. *Let Δ and Δ' be two simplicial fans in $N_{\mathbb{Q}}$ with the same support. Then there exists a simplicial and collapsible cobordism Σ in $N_{\mathbb{Q}}^{+}$ between Δ and Δ'.*

Proof.

The proof consists of several steps. The main idea of Morelli's is to reduce the collapsibility to the projectivity.

Step 1. Show that the projectivity induces the collapsibility.

Proposition 4.4. *Let Σ be a simplicial fan in $N_{\mathbb{Q}}^{+}$ and assume that Σ is a (part of a) projective fan. Then Σ is collapsible.*

Proof.

Since Σ is a part of a projective fan (i.e., a part of a fan Σ' whose corresponding toric variety $X_{\Sigma'}$ is projective), there exists a function $h : \mathrm{Supp}(\Sigma) \to \mathbb{Q}$ which is piecewise linear with respect to the fan Σ and which is strictly convex, i.e., we have

$$\frac{1}{2}\{h(v) + h(u)\} \geq h(\frac{1}{2}\{v + u\})$$

whenever the line segment $\overline{vu}$ is in $\mathrm{Supp}(\Sigma)$ and the strict inequality holds whenever x and y are in two distinct maximal cones (cf.[Fulton][Oda]).

Let σ and σ' be two circuits with a directed edge, i.e., there exists a point $p \in \partial_{+}\overline{Star(\sigma)} \cap \partial_{-}\overline{Star(\sigma')}$ such that

$$p - (0,\epsilon) \in \overline{Star(\sigma)}, p + (0,\epsilon) \in \overline{Star(\sigma')} \text{ for } 0 << \epsilon << 1.$$

Take a maximal π-dependent cone $p \in \eta \supset \sigma$ (resp. $p \in \eta' \supset \sigma'$) of $\overline{Star(\sigma)}$ (resp. of $\overline{Star(\sigma')}$) such that $p - (0,\epsilon) \in \eta$ (resp. $p + (0,\epsilon) \in \eta'$).

Take also linear functions $h_{\eta}, h_{\eta'}, h_{\sigma}, h_{\sigma'}$ which coincide with the restrictions $h|_{\eta}, h|_{\eta'}, h|_{\sigma}, h|_{\sigma'}$, respectively.

Then by the convexity of the function h, setting the coordinates of $p = (x, y)$ we have

$$\frac{1}{2}\{h(x, y + \epsilon) + h(x, y - \epsilon)\} > h(x, y)$$

or equivalently

$$h_{\eta'}(0,1) > h_{\eta}(0,1),$$

and hence

$$h_{\sigma'}(0,1) > h_{\sigma'}(0,1).$$

(Note that $(0,1) \in span_{\mathbb{Q}}(\sigma)$ for any π-dependent cone σ.)

If $\sigma_1, \cdots, \sigma_l$ are circuits determining a directed path in the circuit graph of Σ, then the above observation shows

$$h_{\sigma_1}(0,1) < \cdots < h_{\sigma_l}(0,1).$$

Thus the path cannot be a cycle. Therefore, Σ is collapsible.

Step 2. Show the toric version of Moishezon's theorem.

Theorem 4.5. *Let Σ be a fan in $N_{\mathbb{Q}}^+$. Then there exists a fan $\tilde{\Sigma}$ obtained from Σ by a sequence of star subdivisions such that $\tilde{\Sigma}$ is a (part of a) projective fan.*

Proof.

We may assume that $\mathrm{Supp}(\Sigma) = N_{\mathbb{Q}}^+$ and that Σ is simplicial and nonsingular by applying some appropriate sequence of star subdivisions to the original Σ.

By the toric version of Chow's Lemma (See, e.g., [Oda] §2.3.), we have a projective fan Σ' which is a refinement (not necessarily obtained by a sequence of star subdivisions) of Σ, i.e., we have a projective toric variety $X_{\Sigma'}$ with an equivariant proper birational morphism onto X_Σ

$$g : X_{\Sigma'} \to X_\Sigma.$$

By the toric version of Hironaka's elimination of indeterminacy (See [DeConcini-Procesi].) we can take a fan $\tilde{\Sigma}$ obtained from Σ by a sequence of smooth star subdivisions such that there exists an equivariant proper birational morphism

$$f : X_{\tilde{\Sigma}} \to X_{\Sigma'}.$$

Since $g \circ f$ is projective as it is a sequence of smooth blowups and since g is separated, f is also projective. Now since Σ' is a projective fan, so is $\tilde{\Sigma}$.

Step 3. Composition of (collapsible) cobordisms

Step 2 produces a collapsible and simplicial cobordism between $\partial_-\tilde{\Sigma}$ and $\partial_+\tilde{\Sigma}$, where $\tilde{\Sigma}$ is a (part of a) projective fan and hence collapsible and where $\pi(\partial_-\tilde{\Sigma})$ (resp. $\pi(\partial_+\tilde{\Sigma})$) is obtained from Δ (resp. Δ') by a sequence of star subdivisions. (Or equivalently Δ (resp. Δ') is obtained from $\pi(\partial_-\tilde{\Sigma})$ (resp. $\pi(\partial_+\tilde{\Sigma})$) by a sequence of star assemblings.) We only have to construct a collapsible and simplicial cobordism Σ_Δ between Δ and $\pi(\partial_-\tilde{\Sigma})$ and another $\Sigma_{\Delta'}$ between $\pi(\partial_+\tilde{\Sigma})$ and Δ' so that we compose them together $\Sigma_\Delta \circ \tilde{\Sigma} \circ \Sigma_{\Delta'}$ to obtain a collapsible and simplicial cobordism between Δ and Δ'.

Proposition-Definition 4.6. *Let Σ_1 and Σ_2 be cobordisms in $N_{\mathbb{Q}}^+$ such that*

(o) $\Sigma_1 \cup \Sigma_2$ *is again a fan in* $N_{\mathbb{Q}}^+$,

(i) $\Sigma_1 \cap \Sigma_2 = \partial_+\Sigma_1 \cap \partial_-\Sigma_2$,

(ii) For any cone $\sigma \in \partial_+\Sigma_2$

$$\pi(\sigma) \not\subset \partial\{\pi(\partial_+\Sigma_1 \cup \partial_+\Sigma_2)\} \text{ and } \pi(\sigma) \subset \partial(\pi(\partial_+\Sigma_2)) \Longrightarrow \sigma \in \partial_+\Sigma_1.$$

For any cone $\sigma \in \partial_+\Sigma_1$

$$\pi(\sigma) \not\subset \partial\{\pi(\partial_-\Sigma_1 \cup \partial_-\Sigma_2)\} \text{ and } \pi(\sigma) \subset \partial(\pi(\partial_-\Sigma_1)) \Longrightarrow \sigma \in \partial_-\Sigma_2.$$

Then the union $\Sigma_1 \cup \Sigma_2$, which we call the composition of Σ_1 with Σ_2 and denote by $\Sigma_1 \circ \Sigma_2$, is a cobordism.

Moreover, if both Σ_1 and Σ_2 are simplicial and collapsible, then so is the composition $\Sigma_1 \circ \Sigma_2$.

Proof.

By the condition (o) the composition $\Sigma_1 \circ \Sigma_2$ is a fan. The conditions (i) and (ii) guarantee that $\pi : \partial_-(\Sigma_1 \circ \Sigma_2) \to N_{\mathbb{Q}}$ and $\pi : \partial_+(\Sigma_1 \circ \Sigma_2) \to N_{\mathbb{Q}}$ are isomorphisms of linear complexes onto its images and that the support of $\Sigma_1 \cup \Sigma_2$ lies between the lower face and the upper face. Thus $\Sigma_1 \circ \Sigma_2$ is a cobordism. The "Moreover" part of the assertion is also clear.

Proposition 4.7. *Let $\tilde{\Delta}$ be a simplicial fan in $N_{\mathbb{Q}}$ obtained from another simplicial fan Δ in $N_{\mathbb{Q}}$ by a sequence of star subdivisions and star assemblings. Suppose Δ is embedded in $N_{\mathbb{Q}}$*

$$s : \Delta \hookrightarrow N_{\mathbb{Q}}^+$$

so that $\pi \circ s$ is the identity of the fan.

Then there exists a simplicial and collapsible cobordism Σ between Δ and $\tilde{\Delta}$ (resp. between $\tilde{\Delta}$ and Δ) such that

$$\partial_-\Sigma = s(\Delta)$$
$$(\textit{resp. } \partial_+\Sigma = s(\Delta)).$$

Proof.

We only have to prove the assertion when the sequence consists of a single star subdivision or a star assembling.

Suppose $\tilde{\Delta}$ is obtained from Δ by the star subdivision with respect to a ray ρ passing through the relative interior of a face $\tau \in \Delta$. Say the ray ρ is generated by a primitive vector v_ρ. Then fixing some sufficiently large $y_\rho >> 0$ we only have to take

$$\Sigma = s(\Delta) \cup \{s(\zeta) + (v_\rho, y_\rho); \zeta \in \Delta, \zeta \subset \sigma \text{ for some } \sigma \in \Delta \text{ with } \sigma \ni \rho\}.$$

Suppose $\tilde{\Delta}$ is obtained from Δ by the star assembling, which is the inverse of the star subdivision with respect to a ray ρ passing through the relative interior of a face $\sigma \in \tilde{\Delta}$. Let σ be generated by extremal rays $\rho_1, \cdots, \rho_k$

$$\sigma =< \rho_1, \cdots, \rho_k > .$$

We construct $\Sigma_1, \cdots, \Sigma_k$ with $s_i : \Delta \xrightarrow{\sim} \partial_+ \Sigma_i$ and Σ as required inductively.

Fixing some sufficiently large $y_{\rho_1} >> 0$ we take

$$\Sigma_1 = s(\Delta) \cup \{s(\zeta) + (v_{\rho_1}, y_{\rho_1}); \zeta \in \Delta, \zeta \subset \sigma \text{ for some } \sigma \in \Delta \text{ with } \sigma \ni \rho_1\}.$$

Obviously $\partial_+ \Sigma_1$ is isomorphic to Δ via the projection π, and we set the inverse $s_1 : \Delta \xrightarrow{\sim} \partial_+ \Sigma_1$.

Suppose we have already constructed $\Sigma_1, \cdots, \Sigma_{i-1}$ with $0 << y_{\rho_1} << \cdots << y_{\rho_{i-1}}$ and with the isomorphisms $s_1, \cdots, s_{i-1}$ from Δ to $\partial_+ \Sigma_1, \cdots, \partial_+ \Sigma_{i-1}$. Then we take by fixing some sufficiently large $y_{\rho_i} >> y_{\rho_{i-1}}$

$$\Sigma_i = s_{i-1}(\Delta) \cup \{s_{i-1}(\zeta) + (v_{\rho_i}, y_{\rho_i}); \zeta \in \Delta, \zeta \subset \sigma \text{ for some } \sigma \in \Delta \text{ with } \sigma \ni \rho_i\}.$$

Again clearly $\partial_+ \Sigma_i$ is isomorphic to Δ via the projection π, and we set the inverse $s_i : \Delta \xrightarrow{\sim} \partial_+ \Sigma_i$.

Thus we have constructed $\Sigma_1, \cdots, \Sigma_k$. We only have to set

$$\begin{aligned} \Sigma = \Sigma_k &\cup \{s_k(\nu) + < s_k(\rho_1), \cdots, s_k(\rho_k) >; \nu \in link_{\tilde{\Sigma}} \sigma\} \\ &\cup \{s_k(\nu) + < s_k(\rho_1), \cdots, s_k(\rho_k) > + s_k(\rho); \nu \in link_{\tilde{\Sigma}} \sigma\}. \end{aligned}$$

This completes the proof of Proposition 4.7.

Thus we complete Step 3 and hence the proof of Theorem 4.5.

In §5, starting from a collapsible and simplicial cobordism between two nonsingular fans Δ and Δ' (which we constructed in this section), we try to construct another cobordism which is not only collapsible and simplicial but also π-nonsingular, by further star subdividing the original cobordism. It is worthwhile to note that the collapsibility is preserved under star subdivisions.

Lemma 4.8. *Let Σ be a simplicial cobordism in $N_{\mathbb{Q}}^+$, which is collapsible. Then any simplicial cobordism $\tilde{\Sigma}$ obtained from Σ by a star subdivision, with respect to a ray ρ, is again collapsible.*

Proof.

Note first that if Σ consists of the closed star of a single circuit then $\rho \cdot \Sigma = \rho \cdot \overline{Star(\sigma)}$ is easily seen to be collapsible.

In general, order the circuits of Σ

$$\sigma_1, \sigma_2, \cdots, \sigma_m$$

so that σ_i is minimal among $\sigma_i, \sigma_{i+1}, \cdots, \sigma_m$ according to the partial order given by the circuit graph. Then setting

$$\Sigma = \cup_{i=1}^m \overline{Star(\sigma_i)} \cup \partial_+ \Sigma,$$

we have

$$\rho \cdot \Sigma = \cup_{i=1}^m \rho \cdot \overline{Star(\sigma_i)} \cup \rho \cdot \partial_+ \Sigma$$

and

$$\rho \cdot \Sigma = \{\rho \cdot \overline{Star(\sigma_1)}\} \circ \cdots \circ \{\rho \cdot \overline{Star(\sigma_m)}\} \circ \{\rho \cdot \partial_+ \Sigma\}$$

is collapsible by the first observation and by Proposition-Definition 4.6.

§5. π-Desingularization.

The purpose of this section, which is technically most subtle, is to show the follwoing theorem of "π-desingularization".

Theorem 5.1. *Let Σ be a simplicial fan in $N_{\mathbb{Q}}^{+}$. Then there exists a simplicial fan $\tilde{\Sigma}$ obtained from Σ by a sequence of star subdivisions such that $\tilde{\Sigma}$ is π-nonsingular. Moreover, the sequence can be taken so that any π-independent and already π-nonsingular face of Σ remains unaffected during the process.*

First we describe the outline of Morelli's strategy.

Naively, just LIKE the case of the usual desingularization of toric fans, we would like to subdivide any π-independent face with π-multiplicity bigger than 1 so that its π-multiplicitiy drops. However, UNLIKE the case of the usual desingularization, we might introduce a new π-independent face of uncontrollably high π-multiplicity if we subdivide blindly, though we may succeed in decreasing the π-multiplicity of the π-independent face that we picked originally. This is where the difficulty lies ! We outline Morelli's ingeneous strategy to subdivide carefully to avoid introducing new π-independent faces with high π-multiplicity and achieve π-desingularization. It consists of the following four steps:

Step 1: Introduce the invariant "π-multiplicity profile" π-m.p.(Σ) of a simplicial fan Σ, which measures how far Σ is from being π-nonsingular.

Step 2: Observe that the star subdivision $\eta' = Mid(\tau, l_q) \cdot \eta$ of a simplex η by an interior point of a face τ does not increase the π-multiplicity profile, i.e.,

$$\pi\text{-m.p.}(\eta') \leq \pi\text{-m.p.}(\eta)$$

if

(i) τ is "codefinite" with respect to η, and

(ii) the interior point corresponds to the midray $Mid(\tau, l_q)$, where the ray l_q is generated by a lattice point $q \in par(\pi(\tau))$.

Moreover, if τ is contained in a maximal π-independent face γ of η with the maximum π-multiplicity h_η, i.e., if

$$\tau \subset \gamma \text{ and } \pi\text{-mult}(\gamma) = h_\eta = \max\{\pi\text{-mult}(\zeta); \zeta \subset \eta\},$$

then the π-multiplicity profile strictly drops

$$\pi\text{-m.p.}(\eta') < \pi\text{-m.p.}(\eta).$$

Step 3: Let τ be a π-independent face in the closed star $\overline{Star(\sigma)}$ of a circuit σ in Σ. Introduce the notion of the star-subdivision by the negative or positive center point of σ. We can find Σ° such that

(i) Σ° is obtained by a succession of appropriate star-subdivisions by negative or positive center points of circuits inside of σ,

(ii) the π-multiplicity profile does not increase, i.e.,

$$\pi\text{-m.p.}(\Sigma^\circ) \leq \pi\text{-m.p.}(\Sigma),$$

(iii) τ is a face of Σ° such that τ is codefinite with respect to every cone $\eta \in \Sigma^\circ$ containing τ.

Step 4: Combine Step 2 and Step 3 to find $\tilde{\Sigma}$ obtained from Σ by a succession of star-subdivisions such that the π-multiplicity profile strictly drops

$$\pi\text{-m.p.}(\tilde{\Sigma}) < \pi\text{-m.p.}(\Sigma).$$

As the set of the π-multiplicity profiles satisfies the descending chain condition, we reach a π-nonsingular fan after finitely many star-subdivisions as required.

In fact, by Step 3 we can find a π-independent face τ of a maximal cone $\eta' \subset \Sigma^\circ$ such that

(i) π-m.p.(η') is maximum among the π-multiplicity profiles of all the maximal cones of Σ°,

(ii) τ is contained in a maximal π-independent face γ of η' with the maximum π-multiplicity π-mult$(\gamma) = h_{\eta'}$,

(iii) τ is codefinite with respect to η' and with respect to all the other maximal cones containing τ,

(iv) we can find a lattice point $q \in par(\pi(\tau))$.

We only have to set

$$\tilde{\Sigma} = Mid(\tau, l_q) \cdot \Sigma^\circ$$

to observe by Step 2 that

$$\pi\text{-m.p.}(\tilde{\Sigma}) < \pi\text{-m.p.}(\Sigma).$$

This completes the process of π-desingularization.

Now we discuss the details of each step.

$\boxed{\textit{Step } 1}$

Definition 5.2. *Let γ be a simplicial and π-independent cone in $N^+_{\mathbb{Q}}$. We define the π-multiplicity of γ to be*

$$\pi\text{-}mult(\gamma) = |\det(v_1, \cdots, v_n)|,$$

where the v_i are the primitive vectors of the projections of the generators ρ_i for γ

$$\gamma = < \rho_1, \cdots, \rho_n >$$

with

$$v_i = n(\pi(\rho_i)) \quad i = 1, \cdots, n.$$

Let η be a simplicial cone (i.e., a simplex) in $N^+_{\mathbb{Q}}$ with

$$\begin{aligned} &h_\eta = \max\{\pi\text{-}mult(\gamma); \gamma \text{ is a } \pi\text{-independent face of } \eta\}, \\ &k_\eta = \dim \sigma \text{ where } \sigma \text{ is the unique circuit contained in } \eta, \\ &r_\eta = \text{the number of the maximal } \pi\text{-independent faces of } \eta \\ &\quad \text{having the maximum } \pi\text{-multiplicity } h_\eta. \end{aligned}$$

We define the π-multiplicity profile π-m.p.(η) of η to be the ordered quadruple of numbers

$$\pi\text{-m.p.}(\eta) = (a_\eta, b_\eta, c_\eta, d_\eta)$$

where

$$\begin{aligned}
a_\eta &= h_\eta \\
b_\eta &= \begin{cases} 0 \text{ if } r_\eta \le 1 \\ 1 \text{ if } r_\eta > 1, \end{cases} \\
c_\eta &= \begin{cases} 0 \text{ if } b_\eta = 0 \\ k_\eta \text{ if } b_\eta = 1, \end{cases} \\
d_\eta &= \begin{cases} 0 \text{ if } c_\eta = 0 \\ r_\eta \text{ if } c_\eta > 0. \end{cases}
\end{aligned}$$

We order the set of the π-multiplicity profiles of all the simplicial cones in $N_{\mathbb{Q}}^{+}$ lexicographically.

We define the π-multiplicity profile π-m.p.(Σ) of a simplicial fan Σ in $N_{\mathbb{Q}}^{+}$ to be

$$\pi\text{-m.p.}(\Sigma) = [g_\Sigma; s]$$

where

$$\begin{aligned}
g_\Sigma &= \max\{\pi\text{-m.p.}(\eta); \eta \text{ is a maximal simplicial cone of } \Sigma\}, \\
s &= \text{the number of the maximal simplicial cones of } \Sigma \\
&\quad \text{having the maximum } \pi\text{-multiplicity profile } g_\Sigma.
\end{aligned}$$

When a simplicial fan Σ consists of only one maximal simplicial cone η (and its faces), we understand as a convention

$$\pi\text{-m.p.}(\Sigma) = [\pi\text{-m.p.}(\eta); 1] = \pi\text{-m.p.}(\eta).$$

The definition of the invariant π-multiplicity profile may look heuristic at this point. At the end of the section, we discuss how Morelli reached this definition after a couple of false trials in [Morelli1,2]. The behavior of the π-multiplicity profile under several kinds of star-subdivisions will be the key in Step 3.

$\boxed{\textit{Step } 2}$

Definition 5.3. *A π-independent face τ is codefinite with respect to a π-dependent simplicial cone η iff the set of generators of τ does not contain both positive and negative extremal rays ρ_i of η. That is to say, if $\Sigma r_i v_i = 0$ is the nontrivial linear relation for η among the primitive vectors $v_i = n(\pi(\rho_i))$, then the generators for τ contain only those extremal rays in the set $\{\rho_i; r_i < 0\}$ or in the set $\{\rho_i; r_i > 0\}$, exclusively.*

Notation 5.4. *Let τ be a simplicial cone in a simplicial fan Σ in $N_{\mathbb{Q}}^{+}$ and l a ray in $\pi(\tau)$. Then we define the "midray" $Mid(\tau, l)$ to be the ray generated by the middle point of the line segment $\tau \cap \pi^{-1}(n(l))$. (If $\tau \cap \pi^{-1}(n(l))$ consists of a point, then $Mid(\tau, l)$ is the ray generated by that point.)*

Let γ be a π-independent face in $N_{\mathbb{Q}}$ generated by the extremal rays ρ_i with the corresponding primitive generators $v_1 = n(\pi(\rho_1)), \cdots, v_k = n(\pi(\rho_k)) \in N$. Then define the sets

$$par(\gamma) = \{m \in N; m = \Sigma_i a_i v_i, 0 < a_i < 1\}.$$

Proposition 5.5. *Let τ be a π-independent face of a simplicial cone η. Assume τ is codefinite with respect to η. Let*

$$\eta' = Mid(\tau, l_q) \cdot \eta$$

be the star subdivision of η by the midray $Mid(\tau, l_q)$ where the ray l_q is generated by a lattice point $q \in par(\pi(\tau))$. Then the π-multiplicity profile does not increase under the star-subdivision, i.e.,

$$\pi\text{-m.p.}(\eta') \leq \pi\text{-m.p.}(\eta).$$

Moreover, if τ is contained in a maximal codimension one face γ of η with

$$\pi\text{-}mult(\gamma) = h_\eta = \max\{\pi\text{-}mult(\zeta); \zeta \text{ is a } \pi\text{-independent face of } \eta\},$$

then the π-multiplicity strictly decreases, i.e.,

$$\pi\text{-m.p.}(\eta') \leq \pi\text{-m.p.}(\eta).$$

Proof.

We claim first that all the new maximal π-independent faces γ' of η' have π-multiplicities strictly smaller than h_η, i.e.,

$$\pi\text{-mult.}(\gamma') < h_\eta.$$

Let

$$\rho_1, \cdots, \rho_n$$

be the extremal rays of τ with the corresponding primitive vectors

$$v_i = n(\pi(\rho_i)) \quad i = 1, \cdots, n.$$

We can write

$$0 \neq q = \Sigma_i a_i v_i \quad 0 < a_i < 1$$

as $q \in par(\pi(\tau))$.

Any new maximal π-independent face γ' in η' has the form

$$\gamma' = \rho' + \tau' + \nu$$

where

$$\begin{aligned}\rho' &= Mid(\tau, l_q),\\ \tau' &= \text{a proper face of } \tau \text{ with } \rho' \notin \tau',\\ \nu &\in link_\eta(\tau).\end{aligned}$$

Observe that in general a maximal π-independent face of a simplicial cone in $N_{\mathbb{Q}}^+$ has codimension at most 1 and hence we may assume that in the above expression τ' has codimension at most 2 in τ.

Case: τ' has codimension 1 in τ.

τ' omits, say, ρ_j among the extremal rays of τ. Then

$$\pi\text{-mult}(\rho' + \tau' + \nu') = a_j \cdot \pi\text{-mult}(\tau + \nu) \leq a_j \cdot h_\eta < h_\eta.$$

Case: τ' has codimension 2 in τ.

τ' omits, say, ρ_j and ρ_k among the extremal rays of τ. Observe that in this case $\tau+\nu$ is necessarily π-dependent. In fact, if $\tau+\nu$ is π-independent, then there exists a codimension one face $\tau"(\supset \tau')$ of τ such that we have π-independent faces

$$\tau + \nu \supset \rho' + \tau" + \nu \underset{\neq}{\supset} \rho' + \tau' + \nu,$$

contradicting the maximality of $\rho' + \tau' + \nu$. Let

$$\rho_{n+1}, \cdots, \rho_m$$

be the extremal rays for ν with the corresponding primitive vectors

$$v_i = n(\pi(\rho_i)) \quad i = n+1, \cdots, m$$

as before. Then, since $\tau+\nu$ is π-dependent, we have the nontrivial linear dependence relation

$$\Sigma_{i=1}^m r_i v_i = 0.$$

In order to compute the π-multiplicities, choose a basis of $\{span_{\mathbb{Q}}\pi(\tau+\nu)\} \cap N$. Then

$$\begin{aligned}(\diamond)\quad & \pi\text{-mult}(\rho' + \tau' + \nu)\\ &= |\det(q, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m)|\\ &= |\Sigma_i a_i \cdot \det(v_i, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m)|\\ &= |a_j \cdot \det(v_j, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m) + a_k \cdot \det(v_k, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m)|\end{aligned}$$

On the other hand, we have

$$\begin{aligned}0 &= |\Sigma_i r_i \cdot \det(v_i, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m)|\\ &= |r_j \cdot \det(v_j, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m) + r_k \cdot \det(v_k, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m)|.\end{aligned}$$

Since τ is codefinite with respect to η, either r_j and r_k has the same sign or one of them is 0. (If $r_j = r_k = 0$, then $\tau' + \nu$ would be π-dependent since $\Sigma_{i \neq j,k} r_i v_i = \Sigma_i r_i v_i = 0$. But $\rho' + \tau' + \nu$, containing $\tau' + \nu$, is π-independent, a contradiction!) In the former case, $\det(v_j, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m)$ and $\det(v_k, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m)$ have the opposite signatures and hence continuing the formula ($\diamond$)

$$
\begin{aligned}
&\leq \max\{a_j \cdot |\det(v_j, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m)|, a_k \cdot |\det(v_k, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m)|\} \\
&\leq \max\{a_j \cdot h_\eta, a_k \cdot h_\eta\} \\
&< h_\eta
\end{aligned}
$$

In the latter case (Say, $r_j = 0$ while $r_k \neq 0$.), we have

$$\det(v_k, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m) = 0$$

and hence continuing the formula ($\diamond$)

$$
\begin{aligned}
&= |a_j \cdot \det(v_j, v_1, \cdots, \overset{\vee}{v_j}, \cdots, \overset{\vee}{v_k}, \cdots, v_m)| \leq a_j \cdot h_\eta \\
&< h_\eta.
\end{aligned}
$$

This completes the proof of the claim.

Obesrve that a maximal cone ζ' of η' has the form

$$\zeta' = < \rho', \rho_1, \cdots, \overset{\vee}{\rho_j}, \cdots, \rho_m > \text{ for some } j = 1, \cdots, m.$$

The only possible and old maximal π-independent face of ζ' is $< \rho_1, \cdots, \overset{\vee}{\rho_j}, \cdots, \rho_m >$ and hence the above claim implies

$$\pi\text{-m.p.}(\zeta') \leq (h_\eta, 0, 0, 0).$$

Note that

$$\pi\text{-m.p.}(\eta) \geq (h_\eta, 0, 0, 0)$$

and if the equality holds then there is only one maximal π-independent face $\gamma \subset \eta$ with π-mult$(\gamma) = h_\eta$ and hence we have possibly only one maximal cone ζ' of η', namely the one containing γ, having the π-multiplicity profile equal to $(h_\eta, 0, 0, 0)$. Therefore, we have either

$$\pi\text{-m.p.}(\eta) = (h_\eta, 1, *, *) = [(h_\eta, 1, *, *); 1] \geq [(h_\eta, 0, 0, 0), s] \geq \pi\text{-m.p.}(\eta')$$

or

$$\pi\text{-m.p.}(\eta) = (h_\eta, 0, 0, 0) = [(h_\eta, 0, 0, 0); 1] \geq \pi\text{-m.p.}(\eta').$$

If τ is contained in a maximal codimension one face γ of η with π-mult$(\gamma) = h_\eta$, then the latter case cannot happen and we have the strict inequality.

This completes the proof of Proposition 5.5.

As shown above, the star subdivision by a π-independent face behaves well (choosing an appropriate division point in the interior) if it is codefinite with respect to a π-dependent cone containing it, i.e., if it is codefinite with respect to a circuit in its closed star. In the following, we study how to make a given π-independent face codefinite with respect to circuits in its closed stars, after some specific star-subdivisions.

$\boxed{\textit{Step } 3}$

Let σ be a simplicial cone that is a circuit of dimension k in $N_{\mathbb{Q}}^{+}$

$$\sigma = < \rho_1, \cdots, \rho_k >$$

where the extremal rays ρ_i of σ are generated by $(v_i, w_i) \in N_{\mathbb{Q}}^{+} = N_{\mathbb{Q}} \oplus \mathbb{Q}$ with

$$v_i = n(\pi(\rho_i)) \quad i = 1, \cdots, k$$

being the primitive vectors in N. Let τ be a codimension one face of σ with the maximum π-multiplicity h_σ among all the π-independent faces of σ. Say,

$$\tau = \tau_\alpha = < \rho_1, \cdots, \overset{\vee}{\rho_\alpha}, \cdots, \rho_k > .$$

We have the unique linear dependence relation

$$(\natural) \quad \Sigma_{i=1}^{k} r_i v_i = 0$$

with the conditions

$$\begin{aligned} &|r_\alpha| = 1 \\ &r_1 w_1 + \cdots + r_k w_k > 0. \end{aligned}$$

We note that

$$0 < |r_i| \leq 1 \text{ for } i = 1, \cdots, k$$

where

$$|r_i| = 1 \text{ iff } \pi\text{-mult}(\tau_i) = h_\sigma$$

for

$$\tau_i = < \rho_1, \cdots, \overset{\vee}{\rho_i}, \cdots, \rho_k > .$$

The first inequality $0 < |r_i|$ comes from the fact σ is a circuit and the second inequality and assertion about the equality come from the easy observation

$$\begin{aligned} \pi\text{-mult}(\tau_i) &= \pi\text{-mult} < \rho_1, \cdots, \overset{\vee}{\rho_i}, \cdots, \rho_k > \\ &= |det(v_1, \cdots, \overset{\vee}{v_i}, \cdots, -r_\alpha v_\alpha = \Sigma_{j \neq \alpha} r_j v_j, \cdots, v_k)| \\ &= |r_i| \cdot |det(v_1, \cdots, v_i, \cdots, \overset{\vee}{v_\alpha}, \cdots, v_k)| \\ &= |r_i| \cdot \pi\text{-mult}(\tau) \leq \pi\text{-mult}(\tau). \end{aligned}$$

Thus we conclude that the relation $(\natural)$ is independent of the choice of a codimension one π-independent face τ of σ as long as τ has the maximum π-multiplicity h_σ.

Definition 5.6. *Let σ be a circuit of $N_{\mathbb{Q}}^{+}$ as above. Then the negative (resp. positive) center point $Ctr_{-}(\sigma)$ (resp. $Ctr_{+}(\sigma)$) of σ is defined to be*

$$Ctr_{-}(\sigma) = \Sigma_{r_i<0} v_i$$
$$(\text{resp.} Ctr_{+}(\sigma) = \Sigma_{r_i>0} v_i.)$$

Lemma 5.7. *Let σ be a circuit of $N_{\mathbb{Q}}^{+}$. Then*

$$Ctr_{-}(\sigma), Ctr_{+}(\sigma) \in RelInt(\pi(\sigma)).$$

Proof.

We observe

$$\begin{aligned}
Ctr_{-}(\sigma) &= \Sigma_{r_i<0} v_i \\
&= \Sigma_{r_i<0} v_i + \Sigma_{i=1}^{k} r_i v_i \\
&= \Sigma_{r_i>0} r_i v_i + \Sigma_{r_i<0} (1+r_i) v_i \\
&= (1-\epsilon)\{\Sigma_{r_i<0} v_i\} + \epsilon\{\Sigma_{r_i>0} r_i v_i + \Sigma_{r_i<0}(1+r_i)v_i\} \text{ for } 0 < \epsilon < 1 \\
&= \Sigma_{i=1}^{k} c_i v_i \quad \text{where } c_i = \begin{cases} = \epsilon r_i \text{ when } r_i > 0 \\ = 1 - \epsilon + \epsilon(1+r_i) \text{ when } r_i < 0 \end{cases}
\end{aligned}$$

Since $r_i \neq 0$ and since $r_i \geq -1$ for all i, we see

$$c_i > 0 \quad i = 1, \cdots, k.$$

Thus we conclude

$$Ctr_{-}(\sigma) \in RelInt(\pi(\sigma)).$$

The argument for the statement $Ctr_{+}(\sigma) \in RelInt(\pi(\sigma))$ is identical.

Lemma 5.8. *Let σ be a circuit in $N_{\mathbb{Q}}^{+}$ as above with the negative center point $Ctr_{-}(\sigma)$ (resp. the positive center point $Ctr_{+}(\sigma)$). Let l_{-} (resp. l_{+}) be the ray generated by $Ctr_{-}(\sigma)$ (resp. $Ctr_{+}(\sigma)$) and $\sigma' = Mid(\sigma, l_{-}) \cdot \sigma$ (resp. $\sigma' = Mid(\sigma, l_{+}) \cdot \sigma$) be the subdivision of σ by the midray $Mid(\sigma, l_{-})$ (resp. $Mid(\sigma, l_{+})$). Then every codimension one face ζ of σ with the maximum π-multiplicity h_σ (which stays unchanged through the subdivision and hence can be considered a face $\zeta \in \sigma'$) is codefinite with respect to the (unique) maximal cone in the closed star of ζ in σ'.*

Proof.

We use the same notation as above. We only prove the statement for the negative center as the proof is identical for the positive center.

Observe fisrt that we have

$$Mid(\sigma, l_{-}) \in RelInt(\sigma),$$

since $Ctr_-(\sigma) \in RelInt(\pi(\sigma))$ by the lemma above. Therefore, the star-subdivision with respect to $Mid(\sigma, l_-)$ does not affect ζ, i.e.,

$$\zeta \in \sigma'.$$

Observe secondly (say, $\zeta = \zeta_j =< \rho_1, \cdots, \overset{\vee}{\rho_j}, \cdots, \rho_k >$) that

$$\zeta_j \text{ has maximal } \pi\text{-multiplicity } \iff |r_j| = 1.$$

Case: $r_j = 1$.

In this case, since

$$Ctr_-(\sigma) = \Sigma_{r_i<0} v_i$$

and since the maximal cone σ_j containing ζ_j in σ' is of the form

$$\sigma_j =< Mid(\sigma, l_-), \rho_1, \cdots, \overset{\vee}{\rho_j}, \cdots, \rho_k >,$$

the linear relation for σ_j is given by

$$Ctr_-(\sigma) - \Sigma_{r_i<0} v_i = 0.$$

As ζ_j contains only the extremal rays corresponding to the v_i, which have the same sign (or 0) in the linear relation, ζ_j is codefinite with respect to the (unique) maximal cone σ_j in the closed star of ζ_j in σ'.

Case: $r_j = -1$.

In this case, since

$$Ctr_-(\sigma) = \Sigma_{r_i>0} r_i v_i + \Sigma_{-1<r_i<0} (1 + r_i) v_i$$

and since the maximal cone σ_j containing ζ_j is of the form

$$\sigma_j =< Mid(\sigma, l_-), \rho_1, \cdots, \overset{\vee}{\rho_j}, \cdots, \rho_k >,$$

the linear relation for σ_j is given by

$$Ctr_-(\sigma) - \Sigma_{r_i>0} r_i v_i - \Sigma_{-1<r_i<0} (1 + r_i) v_i = 0.$$

As ζ_j contains only the extremal rays corresponding to the v_i, which have the same sign (or 0) in the linear relation, ζ_j is codefinite with respect to the (unique) maximal cone in the closed star σ_j of ζ_j in σ'.

This completes the proof of Lemma 5.8.

This lemma suggests that we should use the star-subdivision by a neative or positive center of a circuit to achieve codefinireness of a face τ in order to bring the situation of Proposition 5.4 in Step 2. But the lemma only achieves the codefiniteness for a face τ which is contained in a maximal π-independent face with the maximum π-multiplicity but does not analyze the behavior of the π-multiplicity profile. In our process of π-desingularization, we need to achieve codefiniteness for a face τ which is not contanied in a maximal π-independent face with the maximum π-multiplicity and the analysis of the π-multiplicity profile is crucial. Both of these needs are fulfilled by the following proposition, which is at the technical heart of this section.

Proposition 5.9. *Let σ be a circuit of $\dim \sigma > 2$ in $N_{\mathbb{Q}}^{+}$. Then by choosing σ' to be either the star subdivision of σ corresponding to the negative center point or the one by the positive center point, i.e.,*

$$\sigma' = Mid(\sigma, l_{-}) \cdot \sigma \text{ or } Mid(\sigma, l_{+}) \cdot \sigma$$

where l_{-} (resp. l_{+}) is the ray generated by the negative (resp. positive) center point $Ctr_{-}(\sigma)$ (resp. $Ctr_{+}(\sigma)$), we see σ' satisfies one of the following:

A: Every maximal cone δ' of σ' has the π-multiplicity profile strictly smaller than that of σ, i.e.,

$$\pi\text{-m.p.}(\delta') < \pi\text{-m.p.}(\sigma).$$

In particular, we have

$$\pi\text{-m.p.}(\sigma') < \pi\text{-m.p.}(\sigma).$$

B: Every maximal cone δ' of σ', except for one maximal cone κ', has the π-multiplicity profile strictly smaller than that of σ, i.e.,

$$\pi\text{-m.p.}(\delta') < \pi\text{-m.p.}(\sigma)$$

and the exceptional maximal cone κ' has the same π-multiplicity profile as that of σ, i.e.,

$$\pi\text{-m.p.}(\kappa') = \pi\text{-m.p.}(\sigma).$$

In particular, we have

$$\pi\text{-m.p.}(\sigma') = \pi\text{-m.p.}(\sigma).$$

Moreover, there exists a maximal π-independent face γ' of κ' such that

(o) γ' is also a face of σ (i.e., γ' remains untouched under th subdivision),

(i) γ' has the maximum π-multiplicity, i.e.,

$$\pi\text{-mult}(\gamma') = h_{\sigma'} = h_{\sigma},$$

(ii) γ' is codefinite with respect to κ'.

Proof.

Let

$$\sigma = < \rho_1, \cdots, \rho_k >$$

where the extremal rays ρ_i are generated by $(v_i, w_i) \in N_{\mathbb{Q}}^{+}$ with

$$v_i = n(\pi(\rho_i)) \quad i = 1, \cdots, k$$

being the primitive vectors.

Let

$$\Sigma r_i v_i = 0$$

be the nontrivial linear relation so that

$$\Sigma r_i w_i > 0$$

$$|r_i| = 1 \Longleftrightarrow \pi\text{-mult}(\tau_i) = h_{\sigma} \text{ for } \tau_i = < \rho_1, \cdots, \overset{\vee}{\rho_i}, \cdots, \rho_k > .$$

Note that the maximal cones σ'_i of σ' are of the form

$$\sigma'_i =< \rho_0, \rho_1, \cdots, \overset{\vee}{\rho_i}, \cdot\cdot, \rho_k >$$

where ρ_0 is the midray $Mid(\sigma, l_-)$ or $Mid(\sigma, l_+)$ depending on the choice of the negative or positive center point.

We compute the π-multiplicity of the maximal faces τ'_{ij} of σ'_i

$$\tau'_{ij} =< \rho_0, \rho_1, \cdots, \overset{\vee}{\rho_i}, \cdots, \rho_k >$$

as follows: (We let $e \in \mathbb{N}$ be the integer such that $\Sigma_{r_\alpha < 0} v_\alpha = e \cdot n(\pi(\rho_0))$ with $n(\pi(\rho_0))$ being the primitive vector.)

Case of the negative center point: $\rho_0 = Mid(\sigma, l_-)$.

Subcase $r_i > 0$:

$$\begin{aligned}
\pi\text{-mult}(\tau'_{i0}) &= \pi\text{-mult}(\tau_i) \\
\pi\text{-mult}(\tau'_{ij}) &= \frac{1}{e} |\det(\Sigma_{r_\alpha < 0} v_\alpha, v_1, \cdots, \overset{\vee}{v_i}, \cdots, \overset{\vee}{v_j}, \cdots, v_k)| \\
&= \begin{cases} 0 & \text{when } r_j > 0 \\ \frac{1}{e}\pi\text{-mult}(\tau_i) & \text{when } r_j < 0 \end{cases}
\end{aligned}$$

Subcase $r_i < 0$:

$$\begin{aligned}
\pi\text{-mult}(\tau'_{i0}) &= \pi\text{-mult}(\tau_i) \\
\pi\text{-mult}(\tau'_{ij}) &= \frac{1}{e} |\det(\Sigma_{r_\alpha < 0} v_\alpha, v_1, \cdots, \overset{\vee}{v_i}, \cdots, \overset{\vee}{v_j}, \cdots, v_k)| \\
&= \begin{cases} \frac{1}{e}\pi\text{-mult}(\tau_j) & \text{when } r_j > 0 \\ \frac{1}{e}|\pi\text{-mult}(\tau_j) - \pi\text{-mult}(\tau_i)| & \text{when } r_j < 0 \end{cases}
\end{aligned}$$

Symmetrically we compute the other case.

Case of the positive center point: $\rho_0 = Mid(\sigma, l_+)$.

Subcase $r_i < 0$:

$$\begin{aligned}
\pi\text{-mult}(\tau'_{i0}) &= \pi\text{-mult}(\tau_i) \\
\pi\text{-mult}(\tau'_{ij}) &= \frac{1}{e} |\det(\Sigma_{r_\alpha < 0} v_\alpha, v_1, \cdots, \overset{\vee}{v_i}, \cdots, \overset{\vee}{v_j}, \cdots, v_k)| \\
&= \begin{cases} 0 & \text{when } r_j < 0 \\ \frac{1}{e}\pi\text{-mult}(\tau_i) & \text{when } r_j > 0 \end{cases}
\end{aligned}$$

Subcase $r_i > 0$:

$$\begin{aligned}
\pi\text{-mult}(\tau'_{i0}) &= \pi\text{-mult}(\tau_i) \\
\pi\text{-mult}(\tau'_{ij}) &= \frac{1}{e}|\det(\Sigma_{r_\alpha<0} v_\alpha, v_1, \cdots, \overset{\vee}{v_i}, \cdots, \overset{\vee}{v_j}, \cdots, v_k)| \\
&= \begin{cases} \dfrac{1}{e}\pi\text{-mult}(\tau_j) & \text{when } r_j < 0 \\ \dfrac{1}{e}|\pi\text{-mult}(\tau_j) - \pi\text{-mult}(\tau_i)| & \text{when } r_j > 0 \end{cases}
\end{aligned}$$

Using this computation, we can now easily derive the conclusion of the proposition dividing it into the cases according to the cardinalities of the following sets:

$$\begin{aligned}
I^+ &= \{i; r_i > 0\} \\
I^- &= \{i; r_i < 0\} \\
I_1^+ &= \{i; r_i = 1\} \\
I_1^- &= \{i; r_i = -1\}.
\end{aligned}$$

If $e > 1$, then we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$ and hence π-m.p.$(\sigma'_i) < \pi$-m.p.(σ) for all i. Thus we are in the case A.

Therefore, we may assume $e = 1$ in the following.

Case : $\mathbf{2 \leq \#I_1^- \leq \#I_1^+}$

In this case, we choose the negative center point and let $\rho_0 = Mid(\sigma, l_-)$.

When $0 < r_i < 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$ and hence

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $0 < r_i = 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$, $r_{\sigma'_i} \geq \#I_1^- \geq 2$ and hence $b_{\sigma'_i} = 1 = b_\sigma$. Moreover, $c_{\sigma'_i} = k_{\sigma'_i} < k_\sigma = c_\sigma$, since π-mult$(\tau'_{ij}) = 0$ for $j \in I^+ \supset I_1^+$. Thus we have

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $-1 \leq r_i < 0$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$, $r_{\sigma'_i} \geq \#I_1^+ \geq 2$ and hence $b_{\sigma'_i} = 1 = b_\sigma$. We also have $c_{\sigma'_i} = k_{\sigma'_i} \leq k_\sigma = c_\sigma$, since σ is a circuit. Moreover, $d_{\sigma'_i} = r_{\sigma'_i} = r_\sigma - \#I_1^- + 1 < r_\sigma = d_\sigma$, since $\tau'_{ij} = |\pi\text{-mult}(\tau_j) - \pi\text{-mult}(\tau_i)| < h_\sigma$ for $j \in I_1^-$ $j \neq i$. Thus we have

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

Therefore, in this case with the choice of the negative center we conclude we are in Case A and

$$\pi\text{-m.p.}(\sigma') < \pi\text{-m.p.}(\sigma).$$

Case : $\mathbf{1 = \#I_1^- < \#I_1^+}$

In this case, we choose the negative center point and let $\rho_0 = Mid(\sigma, l_-)$.

When $0 < r_i < 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$ and hence

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $0 < r_i = 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$. If $\#I^- = \#I_1^- = 1$, then $r_{\sigma'_i} = 1$ and hence $b_{\sigma'_i} = 0 < b_\sigma = 1$. If $\#I^- > 1$, then $r_{\sigma'_i} > 1$ and hence $b_{\sigma'_i} = b_\sigma$. Moreover, we have $c_{\sigma'_i} = k_{\sigma'_i} < k_\sigma = c_\sigma$, since π-mult$(\tau'_{ij}) = 0$ for $j \in I^+ \supset I_1^+$. Thus we have

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $-1 \leq r_i < 0$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$, $r_{\sigma'_i} \geq \#I_1^+ \geq 2$ and hence $b_{\sigma'_i} = 1$. We also have $c_{\sigma'_i} = k_{\sigma'_i} \leq k_\sigma = c_\sigma$, since σ is a circuit. Moreover, $d_{\sigma'_i} = r_{\sigma'_i} = r_\sigma - \#I_1^- + 1 \leq r_\sigma = d_\sigma$, since $\tau'_{ij} = |\pi\text{-mult}(\tau_j) - \pi\text{-mult}(\tau_i)| < h_\sigma$ for $j \in I_1^-$ $j \neq i$. Thus we have

$$\pi\text{-m.p.}(\sigma'_i) \leq \pi\text{-m.p.}(\sigma).$$

The equality holds only when $r_i = -1$ with i being the sole member of I_1^-, in which case the face τ'_{i0} has the maximum π-multiplicity h_σ and it is codefinite with respect to σ'_i by Lemma 5.8.

Therefore, in this case with the choice of the negative center we conclude we are in Case B and

$$\pi\text{-m.p.}(\sigma') = \pi\text{-m.p.}(\sigma).$$

Case : $\mathbf{1 = \#I_1^- = \#I_1^+ < \#I^+}$

In this case, we choose the negative center point and let $\rho_0 = Mid(\sigma, l_-)$.

When $0 < r_i < 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$ and hence

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $0 < r_i = 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$. If $\#I^- = \#I_1^- = 1$, then $r_{\sigma'_i} = 1$ and hence $b_{\sigma'_i} = 0 < b_\sigma = 1$. If $\#I^- > 1$, then $r_{\sigma'_i} > 1$ and hence $b_{\sigma'_i} = 1 = b_\sigma$. Moreover, we have $c_{\sigma'_i} = k_{\sigma'_i} < k_\sigma = c_\sigma$, since π-mult$(\tau'_{ij}) = 0$ for $j \in I^+ \supset I_1^+$. Thus we have

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $-1 < r_i < 0$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$, $r_{\sigma'_i} = 1$ and hence $b_{\sigma'_i} = 0 < b_\sigma$. Thus we have

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $r_i = 1$, i.e., i is the sole member of I_1^-, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$, $r_{\sigma'_i} = 2$ and hence $b_{\sigma'_i} = b_\sigma$. Moreover, we have $c_{\sigma'_i} = k_{\sigma'_i} = k_\sigma = c_\sigma$ and $r_{\sigma'_i} = 2 = r_\sigma$. Thus we have

$$\pi\text{-m.p.}(\sigma'_i) = \pi\text{-m.p.}(\sigma).$$

The face τ'_{i0} has the maximum π-multiplicity h_σ and it is codefinite with respect to σ'_i by Lemma 5.8.

Therefore, in this case with the choice of the negative center we conclude we are in Case B and

$$\pi\text{-m.p.}(\sigma') = \pi\text{-m.p.}(\sigma).$$

$\mathbf{Case} : \mathbf{0} = \#\mathbf{I}_{\mathbf{1}}^{-} < \mathbf{2} \leq \#\mathbf{I}_{\mathbf{1}}^{+}$

In this case, we choose the positive center point and let $\rho_0 = Mid(\sigma, l_+)$.

When $-1 < r_i < 0$, we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$ and hence

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $0 < r_i < 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$ and hence

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $r_i = 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$, $r_{\sigma'_i} = 1$ and hence $b_{\sigma'_i} - 0 < 1 = b_\sigma$. Thus we have

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

Therefore, in this case with the choice of the positive center we conclude we are in Case A and

$$\pi\text{-m.p.}(\sigma') < \pi\text{-m.p.}(\sigma).$$

$\mathbf{Case} : \mathbf{0} = \#\mathbf{I}_{\mathbf{1}}^{-} < \mathbf{1} = \#\mathbf{I}_{\mathbf{1}}^{+}$

In this case, we choose the positive center point and let $\rho_0 = Mid(\sigma, l_+)$.

When $-1 < r_i < 0$, we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$ and hence

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $0 < r_i < 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$ and hence

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma).$$

When $r_i = 1$, i.e., i is the sole member of I_1^+, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$, $r_{\sigma'_i} = 1$ and hence $b_{\sigma'_i} = 0 = b_\sigma$. Moreover, we have $c_{\sigma'_i} = k_{\sigma'_i} = k_\sigma = c_\sigma$ and $r_{\sigma'_i} = 2 = r_\sigma$. Thus we have

$$\pi\text{-m.p.}(\sigma'_i) = \pi\text{-m.p.}(\sigma).$$

The face τ'_{i0} has the maximum π-multiplicity h_σ and it is codefinite with respect to σ'_i by Lemma 5.8.

Therefore, in this case with the choice of the positive center we conclude we are in Case B and

$$\pi\text{-m.p.}(\sigma') = \pi\text{-m.p.}(\sigma).$$

Symmetrically, we also conclude

Case : $\mathbf{2} \leq \#\mathbf{I}_{\mathbf{1}}^{+} \leq \#\mathbf{I}_{\mathbf{1}}^{-}$

With the choice of positive center point, we are in Case A.

Case : $\mathbf{1} = \#\mathbf{I}_{\mathbf{1}}^{+} < \#\mathbf{I}_{\mathbf{1}}^{-}$

With the choice of the positive center point, we are in Case B.

Case : $\mathbf{1} = \#\mathbf{I}_{\mathbf{1}}^{+} = \#\mathbf{I}_{\mathbf{1}}^{-} < \#\mathbf{I}^{-}$

With the choice of the positive center point, we are in Case B.
Case : $\mathbf{0} = \#\mathbf{I}_{\mathbf{1}}^{+} < \mathbf{2} \leq \#\mathbf{I}_{\mathbf{1}}^{-}$

With the choice of the negative center point, we are in Case A.

Case : $\mathbf{0} = \#\mathbf{I}_{\mathbf{1}}^{+} < \mathbf{1} = \#\mathbf{I}_{\mathbf{1}}^{-}$

With the choice of the negative center point, we are in Case B.

Since the above cases exhaust all the possibilities, we complete the proof for Proposition 5.9.

The next lemma shows that the π-multiplicity of a cone can be computed easily from that of the unique circuit contained in it.

Lemma 5.10. *Let σ be a circuit in Σ and η be a maximal cone in $\overline{Star(\sigma)}$. Then any maximal π-independent face γ of η is of the form*

$$\gamma = \tau + \nu$$

where $\tau = \gamma \cap \sigma$ is a maximal π-independent face of σ and $\nu = link_{\eta}(\sigma)$.

Moreover, there exists $e \in \mathbb{N}$ such that for any γ as above (once η is fixed) we have the formula

$$\pi\text{-}mult(\gamma) = \pi\text{-}mult(\tau) \cdot e.$$

In particular, we have

$$\pi\text{-}m.p.(\eta) = (a_{\eta}, b_{\eta}, c_{\eta}, d_{\eta}) = (e \cdot a_{\sigma}, b_{\sigma}, c_{\sigma}, e \cdot d_{\sigma}).$$

Proof.

Let

$$\sigma = < \rho_1, \cdots, \rho_k >$$

while

$$\eta = < \rho_1, \cdots, \rho_k, \rho_{k+1}, \cdots, \rho_l >$$

where the ρ_i are extremal rays with the coresponding primitive vectors $v_i = n(\pi(\rho_i)) \in N$. Then a maximal π-independent face γ of η is of the form

$$\gamma = < \rho_1, \cdots, \overset{\vee}{\rho_j}, \cdots, \rho_k, \rho_{k+1}, \cdots, \rho_l > = \tau + \nu$$

where

$$\tau = < \rho_1, \cdots, \overset{\vee}{\rho_j}, \cdots, \rho_k > = \gamma \cap \sigma$$

and

$$\nu =< \rho_{k+1}, \cdots, \rho_l >= link_\eta(\sigma).$$

This proves the first assertion.

For "Moreover" part, we have the exact sequence

$$0 \to L \to N_\eta \to Q \to 0$$

where

$$\begin{aligned} L &= span_{\mathbb{Q}}(\pi(\sigma)) \cap N \\ N_\eta &= span_{\mathbb{Q}}(\pi(\eta)) \cap N \end{aligned}$$

and Q is the cokernel, which is torsion free and hence a free $\mathbb{Z}$-module. Take a $\mathbb{Z}$-basis $\{u_1, \cdots, u_{k-1}, u_k, \cdots, u_{l-1}\}$ of N_η so that $\{u_1, \cdot\cdot, u_{k-1}\}$ is a $\mathbb{Z}$-basis of L and $\{u_k, \cdots, u_{l-1}\}$ maps to a $\mathbb{Z}$-basis of Q. With respect to this basis of N_η, the π-multiplicity of γ can be computed

$$\pi\text{-mult}(\gamma) = det \begin{pmatrix} A & B \\ 0 & E \end{pmatrix} = detA \cdot detE = \pi\text{-mult}(\tau) \cdot e,$$

where

$$(v_1, \cdots, \overset{\vee}{v_j}, \cdots, v_k) = \begin{pmatrix} A \\ 0 \end{pmatrix} \quad \& \quad (v_{k+1}, \cdots, v_l) = \begin{pmatrix} B \\ E \end{pmatrix}.$$

This completes the proof of the lemma.

Now it is easy to see the following main consequence of Step 3.

Corollary 5.11. *Let τ be a π-independent face contained in the closed star $\overline{Star(\sigma)}$ of a circuit σ. Then there exists $\{\overline{Star(\sigma)}\}^\circ$ obtained by a succession of star-subdivisions by the negative or positive center points of the circuits (of the intermediate subdivisions) inside of σ such that*

(i) the π-multiplicity profile does not increase, i.e.,

$$\pi\text{-m.p.}(\{\overline{Star(\sigma)}\}^\circ) \leq \pi\text{-m.p.}(\overline{Star(\sigma)}),$$

(ii) τ is a face of $\{\overline{Star(\sigma)}\}^\circ$ and τ is codefinite with respect to every cone $\nu \in \{\overline{Star(\sigma)}\}^\circ$ containing τ.

Proof.

We prove the assertion by induction on the π-multiplicity profile of σ.

If π-m.p.$(\sigma) = (1, 0, 0, 0)$, then by taking $\{\overline{Star(\sigma)}\}^\circ$ to be the star-subdivision corresponding to the negative or positive center point of σ, we see easily that the condition (i) is satisfied, while the condition (ii) is a consequence of Lemma 5.8.

We assume the assertion holds for the case with the π-multiplicity profile smaller than π-m.p.(σ). We take the star-subdivision by the negative or positive center of

σ, according to Proposition 5.8 so that either the case A or the case B holds and hence the π-multiplicity does not increase.

If the case A holds, noting that the circuits of all the maximal cones of the star-subdivision are contained in σ we see the assertion holds immediately by the inductional hypothesis, since all the maximal cones have the π-multiplicity profile strictly smaller than π-m.p.(σ).

Suppose the case B holds. If $\tau \cap \sigma$ is contained in κ', then $\tau \cap \sigma$ is necessarily contained in γ' and hence codefinite with respect to κ'. The other maximal cones have the π-multiplicity profile strictly smaller than π-m.p.(σ) and the assertion again holds by the inductional hypothesis.

This completes the proof of Corollary 5.11 and Step 3.

Now we discuss Step 4.

$\boxed{\textit{Step } 4}$

We start from a simplicial fan Σ.

If Σ is π-nonsingular, then we are done.

So we may assume Σ is not π-nonsingular and hence

$$\pi\text{-m.p.}(\Sigma) = (g_\Sigma; s) \text{ with } g_\Sigma > (1,0,0,0).$$

We only have to construct $\tilde{\Sigma}$ obtained from Σ by a succession of star-subdivisions such that

$$\pi\text{-m.p.}(\tilde{\Sigma}) < \pi\text{-m.p.}(\Sigma).$$

Let η be the maximal cone of Σ such that

$$\pi\text{-m.p.}(\eta) = g_\Sigma$$

with σ being the unique circuit contained in η.

If $\dim \sigma \leq 2$, then we let γ be a maximal π-independent face of η with π-mult$(\gamma) = h_\eta$. We let τ be a minimal π-singular (i.e. not π-nonsingular) face of γ so that we can pick a point $q \in par(\pi(\tau))$.

If $\dim \sigma > 2$, then we take the star-subdivision Σ' of Σ with respect to the negative or positive center point of σ so that either the case A or the case B occurs according to Proposition 5.9.

If the case A occurs, then

$$\pi\text{-m.p.}(\Sigma') < \pi\text{-m.p.}(\Sigma)$$

and we simply have to set $\Sigma^\circ = \Sigma'$.

If the case B occurs, then we take the exceptional cone κ' of σ' with

$$\pi\text{-m.p.}(\kappa') = \pi\text{-m.p.}(\sigma)$$

as described in Proposition 7.8 and take the maximal π-independent face of η such that $\gamma \cap \sigma' = \gamma'$, where γ' is a face of κ' satisfying the conditions (o) and (i) in Proposition 5.9. Observe that by Lemma 5.10 there is a maximal cone η' of Σ' such that

$$\begin{aligned} \eta' \cap \sigma' &= \kappa', \\ \pi\text{-m.p.}(\eta') &= g_{\Sigma'} = g_{\Sigma}, \\ &\gamma \text{ is a face of } \eta' \text{ as well as that of } \eta, \\ \pi\text{-mult}(\gamma) &= h_{\eta'} = h_{\eta}, \text{ and that} \\ &\gamma \text{ is codefinite with respect to } \eta'. \end{aligned}$$

We also take τ to be a minimal π-singular (i.e. not π-nonsingular) face of γ so that we can pick a point $q \in par(\pi(\tau))$.

Now we consider the situation where $\dim \sigma \leq 2$ and the situation where $\dim \sigma > 2$ with the case B together.

Take all the circuits θ' (except for the one contained in κ') of Σ' such that

$$\tau \subset \overline{Star(\theta')}.$$

By Corollary 5.11 of Step 3 for each θ' we can find $\{\overline{Star(\theta')}\}^\circ$ obtained by a succession of star-subdivisions by the negative or positive center points of the circuits (of the intermediate subdivisions) inside of θ' such that the π-multiplicity profile does not increase, i.e.,

$$\pi\text{-m.p.}(\{\overline{Star(\theta')}\}^\circ) \leq \pi\text{-m.p.}(\overline{Star(\theta')}),$$

and that τ is a face of $\{\overline{Star(\theta')}\}^\circ$ and τ is codefinite with respect to every cone $\nu \in \{\overline{Star(\theta)}\}^\circ$ containing τ.

Note that these subdivisions can be carried out simultaneously without affecting each other and that hence we obtain a simplicial fan Σ° obtained from Σ by a successive star-subdivisions such that

(o) the π-multiplicity profile does not increase, i.e.,

$$\pi\text{-m.p.}(\Sigma^\circ) \leq \pi\text{-m.p.}(\Sigma),$$

(i) η' is a maximal cone in Σ° with

$$\pi\text{-m.p.}(\eta') = g_{\Sigma^\circ} = g_{\Sigma'} = g_{\Sigma},$$

(ii) τ is contained in a maximal π-independent face γ of η' with the maximum π-multiplicity $\pi\text{-mult}(\gamma) = h_{\eta'}$,

(iii) τ is codefinite with respect to η' and with respect to all the other maximal cones containing τ,

(iv) we can find a lattice point $q \in par(\pi(\tau))$.

We only have to set
$$\tilde{\Sigma} = \{\tau/\pi l_q\} \cdot \Sigma^\circ$$
to observe by Proposition 5.5 in Step 2 that
$$\pi\text{-m.p.}(\tilde{\Sigma}) < \pi\text{-m.p.}(\Sigma).$$

By the descending chain condition of the set of the π-multiplicity profiles, this completes the process of π-desingularization.

Remark 5.12 in comparison with the original papers [Morelli1,2].

(1) (**Definition of the negative or positive center point**).

The definition of the negative or positive center point $Ctr_-(\sigma), Ctr_+(\sigma)$ as presented here and in [Morelli2] is different from the original definition of the center point $Ctr(\sigma,\tau)$ in [Morelli1]. In spite of the assertions in [Morelli1], $Ctr(\sigma,\tau_j)$ is not alaways in $RelInt(\pi(\tau_j))$, as one can see in some easy examples. This causes a problem in the original argument in [Morelli1], as the subdivision corresponding to the center point may affect not only the cones in the closed star $\overline{Star(\sigma)}$ but also possibly some other cones, which we do not have the controle over. This is the first problematic point in the argument of [Morelli1] noticed by [King2].

(2) (**Definition of the $\pi -$ multiplicity pofile**)

In [Morelli1], the π-multiplicity pofile π-m.p.(η) of a simplicial cone η was defined to be
$$\pi\text{-m.p.}(\eta) = (\pi\text{-mult}(\gamma_1), \cdots, \pi\text{-mult}(\gamma_l))$$
where $\gamma_1, \cdots, \gamma_l$ are the maximal π-independent faces of η with
$$\pi\text{-mult}(\gamma_1) \geq \pi\text{-mult}(\gamma_2) \geq \cdots \geq \pi\text{-mult}(\gamma_l)).$$

Proposition 5.5 holds with this definition, while Proposition 5.9 fails to hold, as [King2] noticed.

(With the slightly coarser definition of the π-multiplicity profile
$$\pi\text{-m.p.}(\eta) = (h_\eta, r_\eta),$$
Proposition 5.5 holds, while Proposition 5.9 fails to hold in a similar way.)

In [Morelli2], the π-multiplicity pofile π-m.p.(η) of a simplicial cone η was changed and defined to be
$$\pi\text{-m.p.}(\eta) = (h_\eta, k_\eta, r_\eta).$$

Proposition 5.9 holds with this definition, while now in turn Proposition 5.5 fails to hold.

The current and correct definition of the π-multiplicity profile, as presented here, was suggested to us by Morelli after we discussed the dilemma as above through e-mail.

(3) (**How to choose τ with q $\in$ par($\pi(\tau)$) and make it codefinite**)

[Morelli1] could be read in such a way that it suggests that for a maximal π-independent face γ with the maximum π-multiplicity $h > 1$ we could take $q \in par(\pi(\gamma))$, which is clearly false in the case $\dim N_{\mathbb{Q}} \geq 3$. The subdivision with respect to $q \in par(\pi(\gamma))$ would only affect the cones in the star $Star\gamma$ and we would only have to analyze those circuits σ such that $\gamma \subset \overline{Star(\sigma)}$. Then the face $\zeta = \gamma \cap \sigma$ has the maximum π-multiplicity h_σ and only Lemma 5.8 would suffice to achieve codefiniteness after the subdivision by the negative or positive center point.

But in general it is only a subface $\tau \subset \gamma$ which contains a point $q \in par(\pi(\tau))$. Now we have to analyze those circuits σ such that $\tau \subset \overline{Star(\sigma)}$ but maybe $\gamma \not\subset \overline{Star(\sigma)}$. Lemma 5.8 is not sufficient any more to achieve the codefiniteness for τ. This is another problematic point in the argument of [Morelli1] noticed also by [King2].

[Morelli2] tries to fix this problem via the use of Proposition 5.9 and what Morelli calls the trivial subdivision of a circuit σ.

Our argument here to achieve Corollary 5.11 solves the problem via the induction on π-multiplicity profile based upon Proposition 5.9 and does not use the trivial subdivision.

§6. The Weak Factorization Theorem

In this section, we harvest the fruit "Weak Factorization Theorem" grown upon the tree of the results of the previous sections.

Proposition 6.1. *We have the weak factorization of a proper equivariant birational map between two nonsingular toric varieties X_Δ and $X_{\Delta'}$ if and only there exists a simplicial, collapsible and π-nonsingular cobordism Σ between the fans Δ and Δ'.*

Proof.

Suppose we have the weak factorization of a proper equivariant birational map between two nonsingular toric varieties X_Δ and $X_{\Delta'}$. Then the fan Δ' is obtained from Δ by a sequence of smooth star subdivisions and smooth star assemblings (in arbitrary order). By Proposition 4.7 there exists a simlicial and collapsible cobordism Σ between Δ and Δ', which is also π-nonsingular by construction (cf. the proof of proposition 4.7).

Conversely, suppose there exists a simplicial, collapsible and π-nonsingular cobordism Σ between the fans Δ and Δ'. Write

$$\Sigma = \cup_\sigma \overline{Star(\sigma)} \cup \partial_+ \Sigma$$

where the union is taken over all the circuits σ. By the collapsibility of Σ, we can order the circuits

$$\sigma_1, \cdots, \sigma_m$$

so that each σ_i is minimal among the circuits $\sigma_i, \sigma_{i+1}, \cdots, \sigma_m$ with respect to the partial order given by the circuit graph of Σ. Accordingly, we have the sequence of fans

$$\begin{aligned}
\Delta = \Delta_0 &= \pi(\partial_- \Sigma) = \pi(\partial_- \{\cup_{i=1}^k \overline{Star(\sigma_i)} \cup \partial_+ \Sigma\}) \\
\Delta_1 &= \pi(\partial_- \{\cup_{i=2}^k \overline{Star(\sigma_i)} \cup \partial_+ \Sigma\}) \\
&\cdots \\
\Delta_j &= \pi(\partial_- \{\cup_{i=j+1}^k \overline{Star(\sigma_i)} \cup \partial_+ \Sigma\}) \\
&\cdots \\
\Delta_k &= \partial_+ \Sigma = \Delta'.
\end{aligned}$$

Note that the fan Δ_{j+1} is obtained from Δ_j by replacing $\partial_- \overline{Star(\sigma_j)}$ with $\partial_+ \overline{Star(\sigma_j)}$, which is the bistellar operation analyzed in §3 and corresponds to a smooth star subdivision followed by a smooth star assemblings. Therefore, we conclude $X_{\delta'}$ is obtained from X_Δ by a sequence of equivariant smooth blowups and smooth blowdowns.

Theorem 6.2 (The Weak Factorization Theorem). *We have the weak factorization for every proper and equivariant birational map between two nonsingular toric varieties X_Δ and $X_{\Delta'}$, i.e., Oda's conjecture holds in the weak from.*

proof.

Let Δ and Δ' be the corresponding two nonsingular fans in $N_{\mathbb{Q}}$ with the same support. Then by Theorem 4.3 there exists a simplicial and collapsible cobordsim Σ in $N_{\mathbb{Q}}^{+}$ between Δ and Δ'. Theorem 5.1 implies there is a simplicial fan $\tilde{\Sigma}$ obtained from Σ by a sequence of star subdivisions such that $\tilde{\Sigma}$ is π-nonsingular and that the process leaves all the π-independent and π-nonsingular cones of Σ unaffected. By Lemma 4.8 we see that $\tilde{\Sigma}$ is also collapsible as well as simplicial and π-nonsingular and that the lower face and upper face of $\tilde{\Sigma}$ are unaffected and hence isomorphic to Δ and Δ', respectively. Thus $\tilde{\Sigma}$ is a simplicial, collapsible and π-nonsingular cobordism between Δ and Δ'. By Proposition 6.1, we have the weak factorization between X_{Δ} and $X_{\Delta'}$. This completes the proof of Theorem 6.2.

§7. The Strong Factorization Theorem.

The purpose of this section is to show the strong factorization theorem, i.e., a proper and equivariant birational map $X_{\Delta'} \dashrightarrow X_{\Delta}$ between smooth toric varieties can be factorized into a sequence of smooth equivariant blowups $X_{\Delta} \leftarrow X_{\Delta''}$ followed immediately by smooth equivariant blowdowns $X_{\Delta''} \to X_{\Delta'}$, based upon the weak factorization theorem (of §6 or [Wlodarczyk]). The main difference between the weak and strong factorization theorems is that the former allows the sequence to consist of blowups and blowdowns in any order for the factorization, while the latter allows the sequence to consist only of blowups first and immediately followed by blowdowns. We should emphasize that this section uses only the statement of the weak factorization theorem and hence is independent of the methods of the previous sections and that the reader, if he wishes, can use [Wlodarczyk]'s result as the starting point for this section (though we continue to phrase the statements in Morelli's terminology that we have been using up to §6).

Our strategy goes as follows. We start with a simplicial, collapsible and π-nonsingular cobordism Σ between Δ and Δ', whose existence is guaranteed by Theorem 6.2. We construct a new cobordism $\tilde{\Sigma}$ from Σ applying an appropriate sequence of star subdivisions such that $\Delta \cong \partial_-\Sigma = \partial_-\tilde{\Sigma}$ is unaffected through the process of the star subdivisions and that the cobordism $\tilde{\Sigma}$ represents, via the bistellar operations, a sequence consisting only of smooth star subdivisions starting from $\partial_-\tilde{\Sigma}$ and ending with $\partial_+\tilde{\Sigma}$. Observing that $\partial_+\tilde{\Sigma}$ is obtained from $\partial_+\Sigma \cong \Delta'$ by a sequence consisting only of smooth star subdivisions, or equivalently $\Delta' \cong \partial_+\Sigma$ is obtained from $\partial_+\tilde{\Sigma}$ by a sequence consisting only of smooth star assemblings, we achieve the strong factorization

$$\Delta \cong \partial_-\Sigma = \partial_-\tilde{\Sigma} \leftrightarrow \partial_+\tilde{\Sigma} \to \partial_+\Sigma \cong \Delta'.$$

First we identify the condition for the bistellar operation to consist of a single smooth star subdivision.

Definition 7.1. *A π-nonsingular simplicial circuit*

$$\sigma =< (v_1, w_1), (v_2, w_2), \cdots, (v_k, w_k) >\in N_{\mathbb{Q}} \oplus \mathbb{Q} = N_{\mathbb{Q}}^+$$

is called pointing up (resp. pointing down) if it has exactly one positive (resp. negative) extremal rays, i.e., we have the linear relation among the primitive vectors v_i (after re-numbering)

$$v_1 - v_2 - \cdots - v_k = 0 \text{ with } w_1 - w_2 - \cdots - w_k > 0$$
$$(\text{resp. } -v_1 + v_2 + \cdots + v_k = 0 \text{ with } -w_1 + w_2 + \cdots + w_k > 0).$$

Lemma 7.2. *Let Σ be a simplicial and π-nonsingular fan in $N_{\mathbb{Q}}^+$ and $\sigma \in \Sigma$ a circuit which is pointing up. Let*

$$\sigma =< (v_1, w_1), (v_2, w_2), \cdots, (v_k, w_k) >\in N_{\mathbb{Q}} \oplus \mathbb{Q} = N_{\mathbb{Q}}^+$$

with the linear relations among the primitive vectors v_i

$$v_1 - v_2 - \cdots - v_k = 0 \text{ with } w_1 - w_2 - \cdots - w_k > 0$$

Then the bistellar operation going from $\partial_-\overline{Star(\sigma)}$ *to* $\partial_+\overline{Star(\sigma)}$ *is a smooth star subdivision with respect to the ray generated by*

$$v_1 = v_2 + \cdots + v_k.$$

If σ *is pointing down with the linear relation*

$$-v_1 + v_2 + \cdots + v_k = 0 \text{ with } -w_1 + w_2 + \cdots + w_k > 0,$$

then the the bistellar operation going from $\partial_-\overline{Star(\sigma)}$ *to* $\partial_+\overline{Star(\sigma)}$ *is a smooth star assembling, the inverse of a smooth star subdivision going from* $\partial_+\overline{Star(\sigma)}$ *to* $\partial_-\overline{Star(\sigma)}$ *with respect to the ray generated by*

$$v_1 = v_2 + \cdots + v_k.$$

Proof.

This is immediate from Theorem 3.2.

Lemma 7.3. *Let* Σ *be a simplicial and* π*-nonsingular fan. Let*

$$\tau =< (v_1, w_1), \cdots, (v_l, w_l) >$$

be a π*-independent face of* Σ *with the* v_i *being the primitive vectors. Let* ρ_τ *be the ray generated by the vector*

$$r(\tau) = (v_1, w_1) + \cdots + (v_l, w_l).$$

If τ *is codefinite with respect to all the circuits* $\sigma \in \Sigma$ *with* $\tau \in \overline{Star(\sigma)}$, *then* $\rho_\tau \cdot \Sigma$ *stays* π*-nonsingular.*

Proof.

Note that though in the statement of Proposition 5.5 the point q was assumed to be taken from $par(\pi(\tau))$, we only need the description

$$q = \Sigma a_i v_i \text{ with } 0 \leq a_i \leq 1$$

(allowing the equality $a_i = 1$) to conclude that the π-multiplicity profile does not increase. Thus we can apply the argument in the proof of Proposition 5.5 with

$$q = v_1 + \cdots + v_l$$

to conclude that π-multiplicity profile does not increase and in particular $Mid(\tau, l_\tau) \cdot \Sigma$ stays π-nonsingular.

Definition 7.4. *Let I be a subset of a fan Σ. Assume I is join closed, i.e., $\tau, \tau' \in I \Longrightarrow \tau + \tau' \in I$ (provided $\tau + \tau' \in \Sigma$). We denote*

$$I \cdot \Sigma = \rho_{\tau_n} \cdots \rho_{\tau_1} \cdot \Sigma$$

where the ρ_{τ_i} are the extremal rays generated by the π-barycenters of the τ_i, i.e., if

$$\tau_i =< (v_1, w_1), \cdots, (v_{l_i}, w_{l_i}) >$$

with the v_j being the primitive vectors, then ρ_{τ_i} is generated by $\Sigma_{j=1}^{l_i}(v_j, w_j)$, and where the τ_i are cones in I ordered so that

$$\dim \tau_i \geq \dim \tau_{i+1} \quad \forall i.$$

(Observe that as I is join closed, $I \cdot \Sigma$ is independent of the choice of the order and it is well-defined.)

The following simple observation of Morelli is the basis of our method in this section.

Lemma 7.5. *Let σ be a circuit in a π-nonsingular fan Σ. Assume σ does not contain a nonzero vertical vector $(0, w) \in N_{\mathbb{Q}}^{+} = N_{\mathbb{Q}} \oplus \mathbb{Q}$ with $w \neq 0$. Let*

$$\sigma =< (v_1, w_1), \cdots, (v_m, w_m), (v_{m+1}, w_{m+1}), \cdots, (v_k, w_k) >,$$

where $v_1, \cdots, v_m, v_{m+1}, \cdots, v_k$ are primitive vectors in N with the unique relation

$$\begin{gathered} v_1 + \cdots + v_m - v_{m+1} - \cdots - v_k = 0 \\ (w_1 + \cdots + w_m - w_{m+1} - \cdots - w_k > 0). \end{gathered}$$

Let

$$\begin{aligned} \sigma_+ &=< (v_1, w_1), \cdots, (v_m, w_m) > \\ \sigma_- &=< v_{m+1}, w_{m+1}), \cdots, (v_k, w_k) > . \end{aligned}$$

(i) The fan $\rho_{\sigma_+} \cdot \overline{Star(\sigma)}$, where ρ_{σ_+} is the extremal ray generated by the π-barycenter $r(\sigma_+) = \Sigma_{i=1}^{m}(v_i, w_i)$ of σ_+, is π-nonsingular and the closed star of a π-nonsingular pointing up circuit σ'.

(ii) If σ is pointing up and I is a join closed subset of σ_-, then $I \cdot \overline{Star(\sigma)}$ is π-nonsingular and the closed star of a π-nonsingular pointing up circuit.

Proof.

(i) First note that by the assumption it is impossible to have all the coefficients in the linear relation to be $+1$ or all to be -1.

Let $\eta \in Star(\sigma)$ be a simplex

$$\eta =< (u_1, w_1'), \cdots, (u_l, w_l'), (v_1, w_1), \cdots, (v_k, w_k) > .$$

Then the maximal cones of $\rho_{\sigma_+} \cdot \eta$ are of the form

$$< (u_1, w_1'), \cdots, (u_l, w_l'), (v_1, w_1), \cdots, \underset{1 \leq i \leq m}{\overset{\vee}{(v_i, w_i)}}, \cdots, (v_m, w_m), (v_{m+1}, w_{m+1}), \cdots, (v_k, w_k), r(\sigma_+) >$$

omitting one of $(v_i, w_i)\ \ 1 \leq i \leq m$ from the generators of σ_+. Therefore,

$$\sigma' =< r(\sigma_+), (v_{m+1}, w_{m+1}), \cdots, (v_k, w_k) >$$

is the unique circuit in $\rho_{\sigma_+} \cdot \overline{Star(\sigma)}$ and

$$\rho_{\sigma_+} \cdot \overline{Star(\sigma)} = \overline{Star(\sigma')}.$$

As ρ_{σ_+} is generated by the vector $r(\sigma_+) = (\Sigma_{i=1}^m v_i, \Sigma_{i=1}^m w_i)$, the unique linear relation for σ' is

$$n(\pi(\rho_{\sigma_+})) - v_{m+1} - \cdots - v_k = 0 \text{ where } n(\pi(\rho_{\sigma_+})) = \Sigma_{i=1}^m v_i$$
$$\text{with } (\Sigma_{i=1}^m w_i) - w_{m+1} - \cdots - w_k > 0.$$

Therefore, the circuit σ' is pointing up. π-nonsingularity is preserved as σ_+ is obviously codefinite with respect to the circuit σ.

(ii) We use the same notation as in (i) with $\sigma_+ =< (v_1, w_1) >$ being the only positive extremal ray of the pointing up circuit σ. Let ζ be the maximal cone in I. Then the maximal cones η' of $\rho_\zeta \cdot \eta$, where ρ_ζ is the generated by the π-barycenter $r(\zeta)$ of ζ, are of the form

$$< (u_1, w_1'), \cdots, (u_l, w_l'), (v_1, w_1), (v_{1+1}, w_{1+1}), \cdots, \underset{(v_j, w_j) \in \zeta.}{\overset{\vee}{(v_j, w_j)}}, \cdots, (v_k, w_k), r(\zeta) >$$

Therefore,

$$\sigma_\zeta =< (v_1, w_1), \text{all the } (v_i, w_i) \quad \notin \zeta, r(\zeta) >$$

is the unique circuit in $\rho_\zeta \cdot \overline{Star(\sigma)}$, which is pointing up with the unique linear relation

$$v_1 - \Sigma_{(v_i, w_i) \notin \zeta} v_i - n(\pi(\rho_\zeta)) = 0 \text{ where } n(\pi(\rho_\zeta)) = \Sigma_{(v_i, w_i) \in \zeta} (v_i, w_i)$$
$$\text{with } w_1 - \Sigma_{(v_i, w_i) \notin \zeta} w_i - (\Sigma_{(v_i, w_i) \in \zeta} w_i) > 0.$$

$\eta \in Star(\sigma)$ being arbitrary, we also have

$$\rho_\zeta \cdot \overline{Star(\sigma)} = \overline{Star(\sigma_\zeta)}.$$

Moreover, every cone in the complement I' of ζ in I is disjoint from σ_ζ. Therefore, σ_ζ is still the unique circuit, which is pointing up, in

$$I \cdot \overline{Star(\sigma)} = I' \cdot \rho_\zeta \cdot \overline{Star(\sigma)}$$

and

$$I \cdot \overline{Star(\sigma)} = \overline{Star(\sigma_\zeta)}.$$

This completes the proof of Lemma 7.5.

The following is an easy consequence of Lemma 7.5.

Lemma 7.6. *Let Σ be a simplicial, collapsible and π-nonsingular cobordism whose circuits are all pointing up and let $I \subset \partial_-\Sigma$ be join closed. Assume the condition $(\star)$:*

$$(\star) \quad I \cap \overline{Star(\sigma)} \subset \{\tau \in \Sigma; \tau \subset \sigma_-\} = \partial_-\sigma \quad \forall \text{ a circuit } \sigma \in \Sigma$$

Then $\Sigma' = I \cdot \Sigma$ is again a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits.

Proof.

By Lemma 4.8 and Lemma 7.3 the cobordism Σ' is again simplicial, collapsible and π-nonsingular. We only have to check that $I \cdot \overline{Star(\sigma)} = (I \cap \overline{Star(\sigma)}) \cdot \overline{Star(\sigma)}$ contains only pointing up circuits for any circuit $\sigma \in \Sigma$, which follows immediately from the condition $(\star)$ and Lemma 7.5 (ii).

Remark 7.7.

Lemma 7.6 is a modification of Lemma 9.7 in [Morelli1] (together with the notion of "neatly founded"), which unfortunately has a counter-example as below. We observe that the notion of "neatly founded" is used only to derive the condition $(\star)$ in the argument of [Morelli1] and carry out our argument here all through with the condition $(\star)$ instead of "neatly foundedness".

Below we recall the definition of "neatly founded" and Lemma 9.7 in [Morelli1] and then present a counter-example.

[Morelli1] defines that Σ is "neatly founded" if for each down definite face $\tau \in \Sigma$ (A face $\tau \in \Sigma$ is down definite iff $\tau \in \partial_-\Sigma$ but $\tau \not\in \partial_+\Sigma$.), there is a circuit $\sigma \in \Sigma$ such that $\tau = \sigma_-$.

Lemma 9.7 in [Morelli1]. *Let Σ be a neatly founded, simplicail, collapsible and π-nonsingular cobordism whose circuits are all pointing up, and let $I \subset \partial_-\Sigma$ be join closed. Then $\Sigma' = I \cdot \Sigma$ is again a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits.*

Counter-Example to Lemma 9.7 in [Morelli1].

We take

$$\begin{aligned}
\rho_1 &= (v_1, 0) \\
\rho_2 &= (v_2, 0) \\
\rho_3 &= (v_3, 0) \\
\rho_4 &= (v_1 + v_2 + v_3, 1) \\
\rho_5 &= (v_1 + v_2 + 2v_3, 2)
\end{aligned}$$

in $N_{\mathbb{Q}}^+ = (N \oplus \mathbb{Z}) \otimes \mathbb{Q} = N_{\mathbb{Q}} \oplus \mathbb{Q}$ with $\dim N_{\mathbb{Q}} = 3$ where v_1, v_2, v_3 form a $\mathbb{Z}$-basis for N. We set Σ to be

$$\Sigma = \left\{ \begin{array}{l} < \rho_1, \rho_2, \rho_3, \rho_4 > \text{ and its faces,} \\ < \rho_2, \rho_3, \rho_4, \rho_5 > \text{ and its faces,} \\ < \rho_1, \rho_3, \rho_4, \rho_5 > \text{ and its faces} \end{array} \right\}.$$

Σ is neatly founded as $< \rho_1, \rho_2, \rho_3 >$ is the only down definite face and there is a circuit $< \rho_1, \rho_2, \rho_3, \rho_4 >$ such that

$$< \rho_1, \rho_2, \rho_3 > = < \rho_1, \rho_2, \rho_3, \rho_4 >_- .$$

All circuits $< \rho_1, \rho_2, \rho_3, \rho_4 >$ and $< \rho_3, \rho_4, \rho_5 >$ are pointing up.

Σ is by construction a simplicial, collapsible and π-nonsingular cobordism between $\Delta = \partial_- \Sigma$ and $\Delta' = \partial_+ \Sigma$.

Take

$$I = \{< \rho_2, \rho_3 > \text{ and its faces}\} .$$

Now Σ and I satisfy all the conditions of Lemma 9.7. On the other hand, $\Sigma' = I \cdot \Sigma$ contains a circuit

$$< \rho_2, M, \rho_4, \rho_5 > \text{ where } M = (v_2 + v_3, 0)$$

which is NOT pointing up!

We resume our proof of the implication the "weak" factorization $\Longrightarrow$ the "strong" factorization.

Proposition 7.8. *Let Σ be a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits. Then there is a simplicial, collapsible and π-nonsingular cobordism Σ' such that*

(o) Σ' contains only pointing up circuits,

(i) Σ' satisfies the condition $(\star)$ for any join closed subset $I \subset \partial_- \Sigma'$,

(ii) Σ' is obtained from Σ by a sequence of star subdivisions, none of which involve $\partial_- \Sigma$, of the π-independent faces which are codefinte with respect to all the circuits.

Proof.

Express the collapsible Σ as

$$\Sigma = \overline{Star(\sigma_m)} \circ \overline{Star(\sigma_{m-1})} \circ \cdots \overline{Star(\sigma_1)} \circ \partial_+ \Sigma$$

for circuits $\sigma_m, \sigma_{m-1}, \cdots, \sigma_1 \in \Sigma$. We prove the lemma by induction on m.

Case $m = 1$: This case is the building block of the construction in the inductional step and we state it in the form of a lemma as below.

Lemma 7.9. *Let Σ be a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits. Let $\overline{Star(\sigma)}$ be the closed star of a circuit $\sigma \in \Sigma$. Let*

$$J = \{\sigma_+ + \nu ; \nu \in link_\Sigma(\sigma)\}.$$

Then

(o) $J \cdot \overline{Star(\sigma)}$ contains only pointing up circuits,

(i) $J \cdot \overline{Star(\sigma)}$ satisfies the condition $(\star)$ for any join closed subset $I \subset \partial_- \{J \cdot \overline{Star(\sigma)}\}$, and

(ii) $J \cdot \overline{Star(\sigma)}$ is obtained from $\overline{Star(\sigma)}$ by a sequence of star subdivisions, none of which involve $\partial_-\overline{Star(\sigma)}$, of the π-independent faces which are codefinite with respect to all the circuits.

Proof.

Let

$$\sigma = < (v_1, w_1), (v_2, w_2), \cdots, (v_k, w_k) >$$

where $v_1, v_2, \cdots, v_k$ are primitive vectors in N satisfying the unique linear relation

$$v_1 - v_2 - \cdots - v_k = 0$$
$$\text{with } w_1 - w_2 - \cdots - w_k > 0.$$

Let $\eta \in Star(\sigma)$ be a simplex

$$\eta = < (u_1, w'_1), \cdots, (u_l, w'_l), (v_1, w_1), \cdots, (v_k, w_k) > .$$

Then the circuits of $J \cdot \eta = \{J \cap \eta\} \cdot \eta$ are the cones of the form

$$\sigma_\nu = < r(\sigma_+ + \nu), (v_2, w_2), \cdots, (v_k, w_k), \text{all the } (u_j, w'_j) \in \nu > \quad \text{for } \nu \in link_\eta(\sigma)$$

(including $\sigma = \sigma_\emptyset = < r(\sigma_+) = (v_1, w_1), (v_2, w_2), \cdots, (v_k, w_k) >$ for $\nu = \emptyset$) satisfying the unique linear relation

$$(v_1 + \Sigma_{(u_j, w'_j) \in \nu} u_j) - v_2 - \cdots - v_k - \Sigma_{(u_j, w'_j) \in \nu} u_j = 0$$
$$\text{with } (w_1 + \Sigma_{(u_j, w'_j) \in \nu} w'_j) - w_2 - \cdots - w_k - \Sigma_{(u_j, w'_j) \in \nu} w'_j > 0.$$

Thus $J \cdot \eta$ contains only pointing up circuits. Since $\eta \in Star(\sigma)$ is arbitrary, we conclude $J \cdot \overline{Star(\sigma)}$ contains only pointing up circuits, proving (o).

We also observe that the maximal cones of $\overline{Star(\sigma_\nu)}$ are of the form

$$< \sigma_\nu, r(\sigma_\nu) + \Sigma_{j=1}^{s} (u_{p(j)}, w'_{p(j)}) \quad s = 1, \cdots, l' = l - \#\{(u_j, w'_j) \in \nu\} >$$

where

$$(u_{p(1)}, w'_{p(1)}), (u_{p(2)}, w'_{p(2)}), \cdots, (u_{p(l')}, w'_{p(l')})$$

are the (u_j, w'_j)'s NOT belonging to ν, ordered in the specified way by a permutation p. Therefore, we conclude that for any join closed subset $I \subset \partial_-\{J \cdot \overline{Star(\sigma)}\}$

$$\begin{aligned} I \cap \overline{Star(\sigma_\nu)} &= I \cap \{\tau \in J \cdot \overline{Star(\sigma)}; \tau \subset \sigma_\nu\} \\ &= \{\tau \in J \cdot \overline{Star(\sigma)}; \tau \subset < I \cap \sigma, \text{all the } (u_j, w'_j) \in \nu >\} \\ &\subset \quad \{\tau \in J \cdot \overline{Star(\sigma)}; \tau \subset (\sigma_\nu)_-\} = \partial_- \sigma_\nu. \end{aligned}$$

Since $\eta \in Star(\sigma)$ is arbitrary, this proves (i).

(ii) is obvious from the construction.

This completes the proof of Lemma 7.9.

We go back to the proof of Proposition 7.8 resuming the induction.

Suppose $m > 1$. Set

$$\Sigma_{m-1} = \overline{Star(\sigma_{m-1})} \circ \cdots \circ \overline{Star(\sigma_1)} \circ \partial_+\Sigma$$

and apply the inductional hypothesis to Σ_{m-1} to obtain Σ'_{m-1} satisfying the conditions (o), (i) and (ii). Then $\overline{Star(\sigma_m)} \circ \Sigma'_{m-1}$ is the result of a sequnce of star subdivisions, none of which involve $\partial_-\Sigma$, of the π-indepenednt faces which are codefinite with respect to all the circuits. Let

$$J = \{(\sigma_m)_+ + \nu; \nu \in link_\Sigma(\sigma_m)\}.$$

We show that $J \cdot (\overline{Star(\sigma_m)} \circ \Sigma'_{m-1})$ satisfies the conditions (o), (i) and (ii).

Since Σ'_{m-1} satisfies (o) and $J \subset \partial_-\Sigma'_{m-1}$ is join closed, the condition $(\star)$ for J with Lemma 7.6 implies that $J \cdot \Sigma'_{m-1}$ is a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits. Lemma 7.9 implies that $J \cdot \overline{Star(\sigma_m)}$ is also a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits. Therefore,

$$\Sigma' = J \cdot (\overline{Star(\sigma_m)} \circ \Sigma'_{m-1}) = (J \cdot \overline{Star(\sigma_m)}) \circ (J \cdot \Sigma'_{m-1})$$

is a simplicial, collapsible and π-nonsingular cobordism satisfying (i).

Observe that

$$\partial_-\Sigma' = \partial_-\overline{Star(\sigma_m)} \cup (\partial_-\Sigma'_{m-1} - RelInt(J)).$$

Thus by construction we have (ii).

Let I be any join closed subset of $\partial_-\Sigma'$. Let $\sigma' \in \Sigma'$ be a circuit. If $\sigma' \in J \cdot \overline{Star(\sigma_m)}$, then by Lemma 7.9 we have

$$I \cap \overline{Star(\sigma')} = (I \cap J \cdot \overline{Star(\sigma_m)}) \cap \overline{Star(\sigma')} \subset \partial_-\sigma'.$$

If $\sigma' \in J \cdot \Sigma'_{m-1}$ and $\sigma' \notin \Sigma'_{m-1}$, then there exists a circuit

$$\sigma =< (v_1, w_1), \cdots, (v_k, w_k) >\in \Sigma'_{m-1}$$

such that

$$\sigma' = \sigma_\zeta =< (v_1, w_1), \text{all the } (v_i, w_i) \quad i \notin \zeta, r(\zeta) >$$

where ζ is the maximal cone in $J \cap \{\tau \in \Sigma'_{m-1}; \tau \subset \sigma\}$, using the same notation as in Lemma 7.5. Observe for any maximal cone $\eta' \in \overline{Star(\sigma')}$ if a face $\tau \subset \eta'$ contains $r(\sigma_m + \nu)$ as one of the generators then $\tau \notin I$. Therefore, by looking at the description of η' in Lemma 7.5 we conclude

$$I \cap \{\tau \subset \Sigma'; \tau \subset \eta'\} = (I \cap \{\tau \in \Sigma'; \tau \subset \sigma'\}) \cap \partial_-\sigma' \subset \partial_-\sigma'.$$

If $\sigma' \in J \cdot \Sigma'_{m-1}$ and also $\sigma' \in \Sigma'_{m-1}$, then the condition $(\star)$ for Σ'_{m-1} implies

$$I \cap \overline{Star(\sigma')} \subset \partial_-\sigma'.$$

Thus we have the condition $(\star)$ proving the condition (i).

This completes the proof of Proposition 7.8.

Theorem 7.10. *Any simplicial, collapsible and π-nonsingular cobordism Σ between Δ and Δ' can be made into a simplicial, collapsible and π-nonsingular cobordism Σ' between Δ and Δ" by a sequence of star subdivisions such that Σ' contains only pointing up circuits and that Δ" is obtained from Δ' by a sequence of smooth star subdivisions.*

Proof.

Express

$$\Sigma = \overline{Star(\sigma_m)} \circ \overline{Star(\sigma_{m-1})} \circ \cdots \circ \overline{Star(\sigma_1)} \circ \partial_+\Sigma.$$

Define a sequence of fans $\tilde{\Sigma}_k, \tilde{\Sigma}'_k$ inductively as follows: Let

$$\tilde{\Sigma}_0 = \tilde{\Sigma}'_0 = \partial_+\Sigma$$
$$\tilde{\Sigma}_k = \rho_{\sigma_k^+} \cdot (\overline{Star(\sigma_k)} \circ \tilde{\Sigma}'_{k-1})$$

where $\tilde{\Sigma}'_{k-1}$ for $k \geq 2$ is obtained from $\tilde{\Sigma}_{k-1}$ by the procedure described in Proposition 7.8 to satisfy the conditions (o), (i) and (ii). (Remark that

$$\partial_-\tilde{\Sigma}_k = \partial_-\tilde{\Sigma}'_k = \partial_-(\overline{Star(\sigma_k)} \circ \overline{Star(\sigma_{k-1})} \circ \cdots \overline{Star(\sigma_1)} \circ \partial_-\Sigma.)$$

Note then that inductively by Lemma 7.5 (i), Proposition 7.8 (i) and Lemma 7.6 $\tilde{\Sigma}_k$ is a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits. Then $\Sigma' = \tilde{\Sigma}_m$ is a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits between $\Delta = \partial_-\Sigma = \partial_-\Sigma'$ and $\Delta" = \partial_+\Sigma'$, which is obtained from Δ' by a sequence of smooth star subdivisions.

This completes the proof of Theorem 7.10.

Corollary 7.10 (The Strong Factorization Theorem). *We have the strong factorization for every proper and equivariant birational map between two nonsingular toric varieties X_Δ and $X_{\Delta'}$, i.e., Oda's conjecture holds in the strong form.*

Proof.

Let Δ and Δ' be the cooresponding two nonsingular fans in $N_{\mathbb{Q}}$ with the same support. Then by Proposition 6.1 and Theorem 6.2 there exists a simplicial, collapsible and π-nonsingular cobordism Σ between Δ and Δ'. By Theorem 7.10 we can make Σ into a simplicial, collapsible and π-nonsingular cobordism Σ' with only pointing up cuircuits between Δ and Δ" such that Δ" is obtained from Δ' by a sequence of smooth star subdivisions. By Lemma 7.2 $\Delta" \cong \partial_+\Sigma'$ is also obtained from $\partial_-\Sigma' \cong \Delta$ by a sequnce od smooth star subdivisions. Thus we have the factorization

$$\Delta \cong \partial_-\Sigma = \partial_-\tilde{\Sigma} \leftrightarrow \partial_+\tilde{\Sigma} \to \partial_+\Sigma \cong \Delta',$$

which corresponds to the strong factorization

$$X_\Delta \leftarrow X_{\Delta"} \to X_{\Delta'}.$$

§8. The Toroidal Case

The purpose of this section is to generalize the main theorem of the previous sections, namely the strong factorization of a proper and equivariant birational map between two nonsingular toric varieties, to the one in the toroidal case.

First we recall several definitions about the toroidal embeddings (cf.[KKMS]) and the notion of a "toroidal" morphism as in [Abramovich-Karu].

Definition 8.1 (Toroidal Embeddings). *Given a normal variety X and an open subset $U_X \subset X$, the embedding $U_X \subset X$ is called toroidal if for every closed point $x \in X$ there exist an affine toric variety X_σ, a closed point $s \in X_\sigma$ and an isomorphism of complete local rings*

$$\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{X_{\sigma,s}}$$

so that the ideal in $\hat{\mathcal{O}}_{X,x}$ generated by the ideal of $X - U_X$ corresponds under this isomorphism to the ideal in $\hat{\mathcal{O}}_{X_{\sigma,s}}$ generated by the ideal of $X_\sigma - T$, where T is the torus.

We will always assume that the irreducible components of $\cup_{i \in I} E_i = X - U_X$ are normal, i.e., $U_X \subset X$ is a toroidal embedding without self-intersection. (In fact, in most of the cases X is nonsingular and $\cup_{i \in I} E_i \subset X$ is a divisor with normal crossings whose irreducible components are all nonsingular.)

The irreducible components of $\cap_{i \in J} E_i$ for $j \subset I$ defines a stratification of X. (These components are the closures of the strata. The closures of the strata formally correspond to the closures of the orbits in local models.

Let S be a stratum in X, which is by definition an open set in an irreducible component of $\cap_{i \in J} E_i$ for some $J \subset I$. The star $Star(S)$ is the union of those strata containing S in their closure (each of them corresponds to some $K \subset J \subset I$). To the stratum S one associates the following data

M^S: the group of Cartier divisors in $Star(S)$ supported in $Star(S) - U_X$

$N^S := Hom(M^S, \mathbb{Z})$

$M^S_+ \subset M^S$: effective Cartier divisors in M^S

$\sigma^S \subset N^S_{\mathbb{R}}$: the dual of M^+_S.

If (X_σ, s) is a local model at $x \in X$ in the stratum S, then

$$\begin{aligned} M^S &\cong M_\sigma / \sigma^\perp \\ N^S &\cong N_\sigma \cap span(\sigma) \\ \sigma^S &\cong \sigma. \end{aligned}$$

The cones glue together to form a conical complex

$$\Delta_X = (|\Delta_X|, \{\sigma^S\}, \{N^S\}),$$

where $|\Delta_X| = \cup_S \sigma^S$ is the support of Δ_X and the lattices N^S form an integral structure on Δ_X with $\sigma^S \hookrightarrow N^S_{\mathbb{R}}$.

Definition 8.2 (A Toroidal Morphism). *A dominant morphism*

$$f:(U_X\subset X)\to(U_Y\subset Y)$$

of toroidal embedding is called toroidal if for every closed point $x\in X$ there exist local models (X_σ,s) at x and $X_{\tau,t}$ at $y=f(x)$ and a toric morphism $g:X_\sigma\to X_\tau$ such that the following diagram commutes

$$\begin{array}{ccc}\hat{\mathcal{O}}_{X,x} & \cong & \hat{\mathcal{O}}_{X_\sigma,s}\\ \uparrow{\scriptstyle \hat{f}^*} & & \uparrow{\scriptstyle \hat{g}^*}\\ \hat{\mathcal{O}}_{Y,y} & \cong & \hat{\mathcal{O}}_{X_\tau,t}.\end{array}$$

Now we can state our main result of this section.

Theorem 8.3. *Let*

$$f:(U_X\subset X)\to(U_Y\subset Y)$$

be a proper birational and toroidal morphism between toroidal embeddings where X and Y are nonsingular and $\cup_{i\in I}E_i=X-U_X$ and $\cup_{j\in J}F_j$ are divisors with normal crossings whose irreducible components are all nonsingular. Then there exist a toroidal embedding (U_V,V) and sequences of blowups with centers being smooth closed strata which factorize f

$$(U_X,X)\leftarrow(U_V,V)\to(U_Y,Y).$$

Lemma 8.4. *Let*

$$f:(U_X\subset X)\to(U_Y\subset Y)$$

be a toroidal morphism between two toroidal embeddings.

(i) f induces a morphism $f_\Delta:\Delta_X\to\Delta_Y$ of complexes such that each $\sigma^S\in\Delta_X$ maps to some $\sigma^{S'}\in\Delta_Y$ linearly

$$f_\Delta:\sigma^S\to\sigma^{S'}$$

with the map of lattices of the integral structures

$$N_{\sigma^S}\to N_{\sigma^{S'}}.$$

(ii) If f is proper and birational, then each $\sigma^S\in\Delta_X$ maps injectively into some $\sigma^{S'}\in\Delta_Y$ linearly

$$f_\Delta:\sigma^S\hookrightarrow\sigma^{S'}$$

and the lattice N_{σ^S} is a saturated sublattice of $N_{\sigma^{S'}}$. In short, Δ_X is a refinement of Δ_Y with $|\Delta_X|=|\Delta_Y|$ preserving the integral structure. Moreover, once we fix the toroidal embedding $(U_Y\subset Y)$, there is a 1-to-1 corespondence between the set of the refinements $f_\Delta:\Delta_X\to\Delta_Y$ preserving the integral structures and the set of toroidal embeddings mapping proper birationally onto $(U_Y\subset Y)$ by toroidal morphisms $f:(U_X\subset X)\to(U_Y\subset Y)$.

Proof.

For a proof, we refer the reader to [KKMS][Abramovich-Karu].

We can reformulate via the lemma our main theorem of this section in terms of the conical complexes (which are always assumed to be finite in this section).

Theorem 8.5. *Let $f_\Delta : \Delta' \to \Delta$ be a map between two nonsingular conical complexes, which is a refinement Δ' of Δ preserving the integral structure. Then there exist a nonsingular conical complex Δ" obtained both from Δ' and from Δ by some sequences of smooth star subdivisions which factorize f_Δ*

$$\Delta \leftarrow \Delta\text{"} \to \Delta.$$

Given a conical comlex Δ, we consider the space $N^S \oplus \mathbb{Z}$, for each $N^S = N_{\sigma^S}$ associated to the cone $\sigma^S \in \Delta$, which can be glued together naturally via the glueing of N^S to form the integral structure. We denote this space $N_\Delta \oplus \mathbb{Z}$. By considering the spaces $(N^S \oplus \mathbb{Z}) \otimes \mathbb{Q}$ and glueing them together, we obtain the space

$$(N_\Delta)^+_{\mathbb{Q}} = (N_\Delta \oplus \mathbb{Z}) \otimes \mathbb{Q} = (N_\Delta)_{\mathbb{Q}} \oplus \mathbb{Q}$$

with the lattices $N^S \oplus \mathbb{Z}$ also glued together to form the integral structure $N_\Delta \oplus \mathbb{Z}$.

If $f_\Delta : \Delta' \to \Delta$ is a refinement of Δ preserving the integral structures, then we can identify $(N_{\Delta'})^+_{\mathbb{Q}}$ with $(N_{\Delta_X})^+_{\mathbb{Q}}$ having the integral structure $N_{\Delta'} \oplus \mathbb{Z} = N_\Delta \oplus \mathbb{Z}$.

Observe that as in the case of toric fans we can define a cobordism Σ in the space $(N_{\Delta_X})^+_{\mathbb{Q}}$ between Δ'_X and Δ as well as the notions of collapsibility, π-nonsingularity, pointing up etc.

Once this is understood, we can carry out the same strategy by Morelli to factorize a proper birational toroidal morphism and we only have to prove

Theorem 8.6. *Let $f_\Delta : \Delta' \to \Delta$ be a map between two nonsingular conical complexes, which is a refinement Δ' of Δ preserving the integral structure. Then there exists a simplicial, collapsible and π-nonsingular cobodism Σ' in $(N_\Delta)^+_{\mathbb{Q}}$ between Δ" and Δ such that Δ" is obtained from Δ' by a sequence of smooth star subdivisions and that Σ' consists only of pointing up circuits and hence Δ" is also obtained from Δ by a sequence of smooth star subdivisions.*

Proof.

We follow exactly the line of argument developed in the previous sections.

First we calim that there exists a simplicial and collapsible cobordism Σ between Δ and Δ'. Recall that in order to construct a cobordism and make it collapsible in the argument for the toric case we have utilized such global theorems as Sumihiro's and Moishezon's, which are no longer applicable in the toroidal case. This calamity can be avoided via the use of the following simple lemma.

Lemma 8.7. *Let Δ be a simplicial conical complex. Then we can embed the barycentric subdivision Δ_B of Δ into a toric fan Δ_B^T in some vector space $N_{\mathbb{Q}}$, i.e., there is a bijective map $i : |\Delta_B| \to |\Delta_B^T| \subset N_{\mathbb{Q}}$ such that it restricts to a linear isomorphism to each cone $i : \sigma \to \sigma^T$. (Note that we do NOT require that i should preserve the integral structure.)*

Proof.

We prove by the induction on the dimension d of Δ and the number k of the cones of the maximal dimension d.

When $d = 1$, i.e., Δ is a finite number of lines, the assertion is obvious.

Suppose the assertion is proved already for a simplicial conical complex of either dimension $< d$ or dimension d with $k-1$ number of the cones of the maximal dimension d. Take a simplicial conical complex Δ of dimension d with k number of the cones of the maximal dimesion d. Choose one cone σ of the maximal dimension d and let $\Delta_\sigma = \Delta - \{\sigma\}$. By the inductional hypothesis, we can embed the barycentric subdivision $(\Delta_\sigma)_B$ into a toric fan $(\Delta_\sigma)_B^T$ in some vector space $N'_{\mathbb{Q}}$

$$i' : |(\Delta_\sigma)_B| \tilde{\to} |(\Delta_\sigma)_B^T| \subset N'_{\mathbb{Q}}.$$

We take $N_{\mathbb{Q}} = N'_{\mathbb{Q}} \oplus \mathbb{Q}$ and regard $N'_{\mathbb{Q}} = N'_{\mathbb{Q}} \oplus \{0\} \subset N_{\mathbb{Q}}$. We only have to take the embedding $i : \Delta_B \to \Delta_B^T$ to be the one such that

$$\begin{aligned} &i|_{(\Delta_\sigma)_B} = i' : |(\Delta_\sigma)_B| \to |(\Delta_\sigma)_B^T| \subset N'_{\mathbb{Q}} \subset N_{\mathbb{Q}} \\ &i(r(\sigma)) = (0,1) \in N'_{\mathbb{Q}} \oplus \mathbb{Q}, \end{aligned}$$

where $r(\sigma)$ is the barycenter of σ and the map i on the cones in Δ_B containing $r(\sigma)$ is defined in the obvious way.

We resume the proof of Theorem 8.6.

Take the barycentric subdivisions Δ'_B and Δ_B of the conical complexes Δ' and Δ, respectively, and let $\tilde{\Delta}_B$ be a simplicial common refinement of Δ'_B and Δ_B.

By Lemma 8.7 we can embed Δ'_B into a toric fan ${\Delta'}_B^T$ in some vector space $N'_{\mathbb{Q}}$. As $\tilde{\Delta}_B$ is a refinement of Δ'_B, it can also be embedded as a toric fan $\tilde{\Delta}_B^T$ in the same space $N'_{\mathbb{Q}}$ by the same map. We can take the star subdivision ${\Delta^\circ}_B^T$ of ${\Delta'}_B^T$ such that it is a refinement of $\tilde{\Delta}_B^T$ (cf. [De Concini-Procesi]). By replacing the original $\tilde{\Delta}_B$ with th pull back of ${\Delta^\circ}_B^T$, we may assume that $\tilde{\Delta}$ is a refinement of Δ_B and Δ'_B and that $\tilde{\Delta}_B$ is obtained from Δ'_B by a sequence of star subdivisions.

By Lemma 8.7 we can embed Δ_B into a toric fan Δ_B^T in some vector space $N_{\mathbb{Q}}$. As $\tilde{\Delta}_B$ is a refinement of Δ_B, it can also be embedded as a toric fan $\tilde{\Delta}_B^T$ in the same space $N_{\mathbb{Q}}$ by the same map. Now we can apply the arguments in §3 and §4 to conclude there is a simplicial and collapsible cobordism in $N_{\mathbb{Q}}^+$ between $(\hat{\Delta_B^T})$ and $(\hat{\tilde{\Delta}_B^T})$, where $(\hat{\Delta_B^T})$ is obtained from Δ_B^T by a sequence of star subdivisions and $(\hat{\tilde{\Delta}_B^T})$ is obtained from $\tilde{\Delta}_B^T$ by another sequence of star subdivisions. We can pull back this cobordism to obtain a simplicial and collapsible cobordism $\tilde{\Sigma}$ in $(N_\Delta)_{\mathbb{Q}}^+$ between $(\hat{\Delta_B})$ and $(\hat{\tilde{\Delta}_B})$, where $(\hat{\Delta_B})$ is obtained from Δ by a sequence of star subdivisions (via the barycentric subdivision Δ_B) and $(\hat{\tilde{\Delta}_B})$ is obtained from Δ' by a sequence of star subdivisions (via the barycentric subdivision Δ'_B and its star subdivision $\tilde{\Delta}_B$). Now we apply Proposition 4.8, which is also valid in the toroidal case, to the lower face $\partial_-\tilde{\Sigma}$ and to the lower face $\partial_+\tilde{\Sigma}$, to extend the cobordism $\tilde{\Sigma}$ to a simplicial and collapsible cobordism Σ between Δ and Δ'.

Now apply the process of π-deingularization described in §5, which is word by word valid also in the toroidal case to make Σ a simplicial, collapsible and π-nonsingular cobordism between Δ and Δ'.

Finally apply the process described in §7, which is again word by word valid in the toroidal case, to the cobordism Σ to obtain the desired simplicial, collapsible and π-nonsingular cobordism Σ' between Δ" and Δ such that Δ" is obtained from Δ' by a sequence of smooth star subdivisions and that Σ' consists only of pointing up circuits and hence Δ" is also obtained from Δ by a sequence of smooth star subdivisions.

This completes the proof of Theorem 8.6 and the verification of the strong factorization theorem for proper birational toroidal morphisms.

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