Choose and complete two of the following three problems (10 points each). Clearly indicate which two you would like scored by (*)'s next to the number. If you do not make any such indication (or mark all three), only the first two will be scored. It is advised that you carefully read all three problems before attempting any of them.

1. Let \( \vec{F}(x, y, z) = (3y^2, 6xy + 8yz, 4y^2) \).

   (a) Compute \( \text{curl} \vec{F} \).

   \[
   \text{Solution:} \text{ Curls are found by crossing } \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \text{ with the original vector. Thus }
   \]
   \[
   \text{curl} \vec{F} = \nabla \times \vec{F} = \left| \begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y^2 & 6xy + 8yz & 4y^2 \end{array} \right|
   = \left( \frac{\partial}{\partial y}(4y^2) - \frac{\partial}{\partial z}(6xy + 8yz), -(\frac{\partial}{\partial x}(4y^2) - \frac{\partial}{\partial z}(3y^2)), \frac{\partial}{\partial x}(6xy + 8yz) - \frac{\partial}{\partial y}(3y^2) \right)
   = (8y - 8y, -(0 - 0), 6y - 6y) = \vec{0}.
   \]

   1 point was given for each component, and 1 point was given for answering with a vector instead of a scalar (remember: curl gives a vector, divergence gives a scalar) for a total of 4 points to this part.

   (b) Does there exist a potential function \( f = f(x, y, z) \) such that \( \vec{F} = \nabla f \)? If so, compute \( f \).

   \[
   \text{Solution: Since curl} \vec{F} = \vec{0}, \vec{F} \text{ is a conservative vector field and thus has a scalar potential function (1 point). Suppose that } f \text{ is the potential. Then } \nabla f = \vec{F}, \text{ so by equating components,}
   \]
   \[
   \frac{\partial f}{\partial x} = 3y^2 \quad (1)
   \frac{\partial f}{\partial y} = 6xy + 8yz \quad (2)
   \frac{\partial f}{\partial z} = 4y^2 \quad (3)
   \]
Integrating (1) with respect to \( x \), our intial guess is that \( f(x, y, z) = 3xy^2 + g(y, z) \), where \( g \) is some function of \( y \) and \( z \). Differentiating our guess with respect to \( y \) results in

\[
\frac{\partial f}{\partial y} = 6xy + \frac{\partial g}{\partial y} = 6xy + 8yz
\]

by (2). Therefore \( \frac{\partial g}{\partial y} = 8yz \), and so integrating this with respect to \( y \) gives \( g(y, z) = 4y^2z + h(z) \), where \( h \) is some function of \( z \). Our guess becomes \( f(x, y, z) = 3xy^2 + 4y^2z + h(z) \). Differentiating this new guess with respect to \( z \),

\[
\frac{\partial f}{\partial z} = 4y^2 + h'(z) = 4y^2
\]

by (3). Therefore \( h'(z) = 0 \), so \( h \) is a constant, which we can take to be zero. 1 point was given for 3\( xy^2 \) and another for 4\( y^2 \).

(c) Let \( C \) be any curve from \((0, 0, 0)\) to \((1, 1, 1)\) and find \( \int_C \vec{F} \cdot d\vec{r} \).

**Solution:** According to the fundamental theorem for line integrals,

\[
\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1, 1, 1) - f(0, 0, 0) = 3 + 4 - 0 = 7.
\]

Two points were given for at least knowing/writing the fundamental theorem; the last point for this part was for the correct final answer (based on your answer from part (b)).

Some people parameterized the straight-line path and did it out directly – this was fine since the vector field \( \vec{F} \) was conservative, but using the fundamental theorem was the intended solution since it is the quickest and uses the information already derived.

2. Finish the following statement (Green’s Theorem): If \( C \) is a positively oriented (counterclockwise), piecewise-smooth, simple closed curve in the \( xy \)-plane, and \( D \) is the region enclosed by \( C \), then

\[
\oint_C P \, dx + Q \, dy = ?
\]

Use this to compute \( \int_C y^3 \, dx - x^3 \, dy \), where \( C \) is the circle of radius 1 centered at the origin, oriented clockwise.

**Solution:** The ?-mark should read \( \iint_D (Q_x - P_y) \, dA \) (this was worth 5 points total – 2 for \( Q_x \), 2 for \( P_y \), and 1 point for the correct signs on each).

For the given integral, we have \( P = y^3 \), \( Q = -x^3 \), so \( Q_x = -3x^2 \) and \( P_y = 3y^2 \). Because the curve is clockwise instead of counterclockwise, Green’s theorem tells us that

\[
\oint_C P \, dx + Q \, dy = -\iint_D -3x^2 - 3y^2 \, dA,
\]

where \( C \) is the unit circle and \( D \) is the unit disk in the \( xy \)-plane (note the extra negative). Converting to polar coordinates, the integral becomes

\[
3 \int_0^{2\pi} \int_0^1 r^2 r \, dr \, d\theta = 6\pi \int_0^1 r^3 \, dr = \frac{6\pi}{4} = \frac{3\pi}{2}.
\]
3. Let \( \vec{F}(x, y, z) = \langle \ln x, \ln(xy), \ln(xyz) \rangle \). Find curl \( \vec{F} \) and div \( \vec{F} \) at the point \((x, y, z) = (1, -1, 3)\).

**curl \( \vec{F} \):**

A. \( \langle -1, 1, 1 \rangle \)  
B. \( \langle 1, 1, 1 \rangle \)  
C. \( \langle -1, -1, 1 \rangle \)  
D. \( \langle 1, -1, 1 \rangle \)  
E. \( \langle -3, -1, 1 \rangle \)  

**div \( \vec{F} \):**

A. 3  
B. 1  
C. \( -\frac{1}{3} \)  
D. -3  
E. \( \frac{1}{3} \)  

**Solution:** Direct computation shows

\[
\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln x & \ln(xy) & \ln(xyz) \end{vmatrix} \\
= \left\langle \frac{1}{y} - 0, -\frac{1}{x} - 0, \frac{1}{x} - 0 \right\rangle \\
= \left\langle \frac{1}{y}, -\frac{1}{x}, \frac{1}{x} \right\rangle \\
\text{div } \vec{F} = \frac{\partial}{\partial x} (\ln x) + \frac{\partial}{\partial y} (\ln xy) + \frac{\partial}{\partial z} (\ln xyz) \\
= \frac{1}{x} + \frac{1}{xy} (x) + \frac{1}{xyz} (xy) \\
= \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.
\]

Plugging in \((x, y, z) = (1, -1, 3)\),

\[
\text{curl } \vec{F} = \left\langle \frac{1}{-1}, -\frac{1}{1}, \frac{1}{1} \right\rangle = \langle -1, -1, 1 \rangle \\
\text{div } \vec{F} = \frac{1}{1} + \frac{1}{-1} + \frac{1}{3} = \frac{1}{3}.
\]

These correspond to answers C and E, for the curl and divergence, respectively.