Chi Li's Research Statement

The existence of canonical metrics are basic questions in geometry. Classical examples are constant Gauss curvature Riemannian metrics on Riemann surfaces. Thurston's geometrization program could also be thought of as finding canonical metrics on three manifolds. More generally we would like to find Einstein metrics on higher dimensional manifolds. When the underlying manifolds are Kähler, the natural candidate canonical metrics are Kähler-Einstein metrics.

Thanks to the work of Yau and Aubin, Kähler-Einstein metrics with non-positive Ricci curvatures can be solved once appropriate cohomological conditions are satisfied. On the other hand, there are complicated obstructions to the existence of Kähler-Einstein metrics with positive Ricci curvatures, which are related to algebraic stability conditions of underlying varieties. Most of my research is centered around the subjects of Kähler-Einstein (KE) metrics and related problems/applications. This is a vast subject which connects several fields of geometry and analysis, including differential geometry, algebraic geometry, pluripotential theory and partial differential equations. My works often combine or connect analytic and algebraic aspects of different subjects, and hence I need to use and develop tools in both analysis and algebra.

I will discuss my work in the order roughly from the most recent to older ones, and along the way propose a few related problems to be studied. I will first focus on the two major projects that I developed after coming to Purdue.

1 Yau-Tian-Donaldson conjecture for singular Fano varieties

The study of Kähler-Einstein metrics on Fano varieties has seen enormous progress in the past decades. The famous Yau-Tian-Donaldson (YTD) conjecture states that a Fano manifold admits a Kähler-Einstein metric if and only if it is K-polystable. A Fano variety is a projective variety with an ample anti-canonical class. The equation for the Kähler-Einstein metric is equivalent to a complex Monge-Ampère equation:

$$Ric(\omega) = \omega \quad \Longleftrightarrow \quad (\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{-\varphi}\Omega. \tag{1}$$

The notion of K-(poly)stability, as introduced by Tian and reformulated by Donaldson, is a purely algebraic condition on the Fano variety and can be deeply studied by using tools from algebraic geometry (see section 3). Due to the work of Tian, Berman, Chen-Donaldson-Sun (CDS) and many others, the Yau-Tian-Donaldson conjecture is now a theorem.

The original approach to prove the existence part of this conjecture is to run some continuity method to solve the equation (1). The difficulty lies in understanding the singularities that one encounters if the above equation is not solvable. In this respect the proof depends on compactness theory developed by Cheeger-Colding-Tian (CCT). Roughly speaking, this means that along some continuity method with a parameter t, the metric space (X, ω_t) converges in the Gromov-Hausdorff topology to some limit metric space $(X_\infty, \omega_\infty)$. In order to understand the regularity of the limit X_∞, one needs some type of partial C^0-estimates as originally proposed by Tian and first established by Donaldson-Sun in high dimensions. For example I showed in my thesis [1] by using Skoda-Siu's method that the truth of partial C^0-estimates implies the effective (w.r.t. metric) finite generation of anti-canonical-rings, which could imply that the Gromov-Hausdorff limit is a projective variety.

Fano varieties are important building blocks of algebraic varieties. Indeed, when a Minimal Model Program ends with a Mori fibration, the fibres are Fano varieties. However smooth Fano manifolds only form a small sub-class of Fano varieties. Since 80's it has been known that the number of deformation classes of smooth Fano varieties in a fixed dimension is finite. The study of singular Fano varieties has been an important subject in algebraic geometry. Three years ago, in his Fields medal work, Birkar proved the boundedness of ϵ-Klt Fano varieties (the so-called BAB conjecture).

A $\mathbb{Q}$-Fano variety will mean a Fano variety with at worst Klt singularities. In a series of works, the author, together with G. Tian and F. Wang, established versions of Yau-Tian-Donaldson conjecture for singular $\mathbb{Q}$-Fano varieties. It is hard to directly adapt the proof of CDS-Tian because metric geometry on singular varieties are poorly understood. So significant new works have to be done. Our first result is:

Theorem 1.1 ([22]). *If a K-polystable $\mathbb{Q}$-Fano variety admits a log resolution such that the discrepancies of all exceptional divisors are non-positive, then it admits a Kähler-Einstein metric.*

This result generalizes the result of CDS-Tian. The idea of proof is natural. We work over a log resolution of the singular variety and, after perturbing the degenerate Kähler class, solve for KE metrics with edge cone singularities (will just be called conical KE). The assumption on discrepancies is used to guarantee that we have a good theory of Kähler metrics with edge cone singularities. However, there are many new technical difficulties to overcome. The first one is to get KE on the resolution, which requires us to derive K-stability of log pairs on the resolution from the K-stability of the original X. Interestingly, its proof depends on the valuative criterions for K-stability that Fujita and I developed ([12]) (see section 3). Then we can use a log version of YTD to get conical KEs on the resolution. In order to get the KE on the original variety, we need to further prove uniform estimates for these conical KEs. This part is quite involved, depending on various techniques including pluripotential theory, CCT-theory and partial C^0-estimates for conical KEs with varying Kähler classes. However currently it is not known how to use such kind of metric approach to prove the YTD without the assumption on log discrepancies.

In 2015 Berman-Boucksom-Jonsson (BBJ) proposed a variational approach to prove the uniform version of Yau-Tian-Donaldson conjecture. This approach is independent of and quite different from that of CDS-Tian, and avoids the use of CCT-theory and partial C^0-estimates. First it was known that the existence of KE is equivalent to the coercivity of Mabuchi energy over the space of Kähler metrics, while K-stability is equivalent to the coercivity of the energy along algebraic rays. Assuming on the contrary that the energy is not coercive, BBJ used the (Archimedean and non-Archimedean) pluripotential theory and algebraic geometry (multiplier ideals) to construct a sequence of destabilizing objects which contradicts the K-stability. However their work only covers the case of smooth Fano manifolds with discrete automorphism groups. Our most recent works, based some new ideas, have greatly extended their approach.

Because a key step in BBJ's argument uses Demailly's approximation result about multiplier ideals which fails in the singular case, they proposed to find a replacement of this result on singular varieties. In early 2019, we realized that there is a way to get around this difficulty by working over the resolution of singular varieties and proving some uniform estimates and convergence results by refining our previous joint work. Although the idea of working on the resolution is natural when dealing with singular varieties, previously it was not expected to work in general since the relevant twisting divisor on the resolution is in-effective. However, I realized that this is not a problem if, roughly speaking, we work more in the non-Archimedean category. As a consequence, we proved:

Theorem 1.2 ([25]). *A $\mathbb{Q}$-Fano variety X with a discrete automorphism group admits a Kähler-Einstein metric if and only if it is uniformly K-stable.*

The key contribution of this work is overcoming the barrier of singularities and removing the assumption on log discrepancies compared to Theorem 1.1. It is a related interesting problem to study twisted KEs with non-effective twisting which includes conical case:

Problem 1.1. *Study the existence of conical KE metrics with cones angles bigger than 2π. Find the correct stability conditions for the existence and study their geometries.*

More recently, I proved the equivariantly uniform version of Yau-Tian-Donaldson conjecture for arbitrary singular $\mathbb{Q}$-Fano varieties. This is the first known result in this generality. To achieve this, I still have to overcome the difficulty caused by continuous automorphism groups. Earlier T. Hisamoto has done some nice works in this direction. He introduced an equivariantly uniform K-stability which corresponds to an analytic criterion obtained from Darvas-Rubinstein's work.

However previously in BBJ's approach there is no clear way to deal with the difficulty caused by continuous automorphism groups. After some effort, I managed to find a solution to this problem, which depends on a new valuative criterion for equivariantly uniform K-stability that generalizes the valuative criterion for uniform K-stability by K. Fujita (see Theorem 3.2). Moreover I need to refine our work in [25] to design much more delicate approximation arguments. As a consequence, I am able to prove the following result (which is actually proved in the more general log Fano case):

Theorem 1.3 ([26]). *A $\mathbb{Q}$-Fano variety X admits a Kähler-Einstein metric if and only if X is $\mathbb{G}$-uniformly K-stable where $\mathbb{G}$ is any connected subgroup of $\mathrm{Aut}(X)_0$ containing a maximal torus.*

As said before, this is the first proved versions of Yau-Tian-Donaldson conjecture for arbitrary possibly singular Fano varieties. Moreover even in the case when X is smooth, my argument gives an alternative proof of the equivariantly uniform version Yau-Tian-Donaldson conjecture compared to CDS-Tian and Datar-Székelyhidi. In practice, this result can be used to obtain Kähler-Einstein metrics on (singular) Fano varieties with large symmetry groups.

Related development and perspectives:

The Kähler-Einstein metrics on singular varieties in the above discussion are usually defined in the sense of pluripotential theory. Understanding in more detail the behavior of singular Kähler-Einstein metrics, or more general Kähler metrics on singular varieties, is a natural and important question. For example the famous Cheeger-Goresky-MacPherson conjecture asks whether L^2-cohomology of singular projective varieties under the suitable Kähler metrics coincides with the intersection cohomology. Currently it is in general quite difficult except in situations where good local models can be found. This regularity question is related to the study of metric tangent cones as we will discuss in section 2.1. We plan to study some related questions in future.

Moreover we hope to extend our study of Yau-Tian-Donaldson conjecture to other canonical metrics. For example there is a YTD conjecture for constant scalar curvature Kähler (CSCK) metrics and for more general extremal metrics. More broadly, the study of canonical metrics can often be put into a general framework involving moment maps associated to holomorphic actions by gauge groups on infinite dimensional Kähler manifolds. This framework, as advocated by Atiyah, Bott, Donaldson etc., contains a lot of problems, including CSCK problem, Hermitian-Einstein metrics and many other nonlinear PDE's coming from geometry and math-physics. However different problems requires different techniques to deal with. There is hope to adapt the variational approach in the study of KE problem to some other problems. But the technical difficulties seem severer and we definitely need more understanding of regularity of destabilizing objects in order to solve them.

2 Normalized volumes

Motivated by the study of metric tangent cones (see section 2.1) and results from Sasaki-Einstein geometry, I introduced a new study of volumes for Klt singularities. I will sketch basic elements in this theory which connects several subjects in complex algebraic geometry.

A Klt singularity is a $\mathbb{Q}$-Gorenstein singularity that has a log resolution whose exceptional divisors have positive log discrepancies. Klt singularities form an important class of singularities in birational algebraic geometry. They are in some sense local analogues of Fano varieties and are preserved under the Minimal Model Program (MMP).

For any germ of klt singularity (X, x), let $\mathrm{Val}_{X,x}$ denote the space of real valuations whose centers over X are x. In [13] I introduced a functional on $\mathrm{Val}_{X,x}$ as follows: For any $v \in \mathrm{Val}_{X,x}$, set:

$$\widehat{\mathrm{vol}}(v) := \widehat{\mathrm{vol}}_X(v) = \begin{cases} A_X(v)^n \mathrm{vol}(v), & \text{if} \quad A_X(v) < +\infty; \\ +\infty, & \text{if} \quad A_X(v) = +\infty. \end{cases}$$

Here the function $A_X(v)$ generalizes the log discrepancy function for divisorial valuations as studied by Mustaţă-Jonsson and Boucksom-de-Fernex-Favre-Urbinati, and $\mathrm{vol}(v)$ is the volume of valuation $v \in \mathrm{Val}_{X,x}$ introduced by Ein-Lazarsfeld-Smith. As a starting point, I proved a uniform estimate:

Proposition 2.1 ([13]). *Let $\mathfrak{m}_x$ denote the maximal ideal of $x \in X$. There exists $C > 0$ depending only on (X, x) such that for any $v \in \mathrm{Val}_{X,x}$,*

$$\widehat{\mathrm{vol}}(v) \geq C \frac{A_X(v)}{v(\mathfrak{m}_x)} \geq C \cdot \mathrm{lct}_X(\mathfrak{m}_x) > 0, \tag{2}$$

where lct_X denotes the log canonical threshold of ideals. In particular $\widehat{\mathrm{vol}}$ has a positive lower bound.

As a consequence of this estimate, we can define a new positively valued invariant for any Klt singularity:

$$\widehat{\mathrm{vol}}(x, X) = \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}_X(v). \tag{3}$$

Moreover it is natural to study the minimization problem of $\widehat{\mathrm{vol}}$ over $\mathrm{Val}_{x,X}$. In [13] I made the following purely algebraic conjecture which was later refined in [16].

Conjecture 2.1 (Stable Degeneration Conjecture, [13, 16]). *Given any Klt singularity $x \in X = \mathrm{Spec}(R)$, $\widehat{\mathrm{vol}}$ has unique minimizer v_* up to rescaling. Furthermore, v_* is quasi-monomial, with a finitely generated associated graded ring $R_0 =_{\mathrm{defn}} \mathrm{gr}_{v_*}(R)$, and the central fibre of the induced degeneration $(X_0 = \mathrm{Spec}(R_0), \xi_{v_*})$ is a K-semistable Fano cone singularity.*

In [13], I realized that there is a connection between this minimization problem with the following inequality proved by de-Fernex-Mustaţă-Ein which was motivated from a completely different subject (birational rigidity of Fano varieties):

$$\mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) \geq n^n, \tag{4}$$

for every $\mathfrak{m}_0$-primary ideal $\mathfrak{a}$ on $\mathbb{C}^n$. I used (4) to show that $\widehat{\mathrm{vol}}(0, \mathbb{C}^n) = n^n$. Soon after my work Yuchen Liu proved that the following general identity (the last identity is by Jonsson-Mustaţă):

$$\inf_{v \in \mathrm{Val}_{X,x}} A_X(v)^n \cdot \mathrm{vol}(v) = \inf_{\mathfrak{a}:\mathfrak{m}_x-\mathrm{primary}} \mathrm{lct}(\mathfrak{a})^n \cdot \mathrm{mult}(\mathfrak{a}) = \inf_{\mathfrak{a}_\bullet} \mathrm{lct}(\mathfrak{a}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{a}_\bullet). \tag{5}$$

In following papers, I continued to established the connection between volume minimization and K-semistability of $\mathbb{Q}$-Fano varieties, and more generally the K-semistability of Fano cones (which was introduced by Collins-Székelyhidi to study Sasaki-Einstein metrics).

Theorem 2.1 ([12, 14, 16, 17]). *On a Fano orbifold cone $X := C((S, \Delta), L)$, ord_S is a minimizer of $\widehat{\mathrm{vol}}_{X,x}$ if and only if (S, Δ) is log-K-semistable. More generally, a Fano cone (X, ξ_0) is K-semistable if and only if wt_{ξ_0} is a minimizer of $\widehat{\mathrm{vol}}$.*

The above theorem strengthens/generalizes the minimization result by Martelli-Sparks-Yau who essentially considered valuations from torus actions. The latter was the main inspiration for me to introduce the above more general minimization problem.

Many results have been proved toward this conjecture 2.1, which concerns the existence, uniqueness and regularity of minimizers of normalized volume functional. In particular, H. Blum proved the existence of minimizing valuations with the help of estimate (2). Divisorial minimizers are well-understood by my joint work with Chenyang Xu (and also some independent work of H. Blum).

Theorem 2.2 ([16]). *Divisorial minimizers are unique (up to rescaling) and come from exceptional divisors of plt blow-ups. In particular, they have finitely generated associated graded rings.*

To prove this result, we studied the change of normalized volumes of models under the (local) MMP. This is a local analogue of the global theory on the behavior of CM weight under the (relative) MMP that we studied several years ago (see section 3).

In general minimizers are not necessarily divisorial. The conjecture on the quasi-monomial property of minimizing valuations turns out to be quite natural. From inequality (5), one knows that if v_* minimizes $\widehat{\mathrm{vol}}_{X,x}$, then v_* calculates the log canonical threshold $\mathrm{lct}(\mathfrak{a}_\bullet(v))$ of valuative ideals. This observation connects the Conjecture 2.1 and Jonsson-Mustaţă's conjecture:

Conjecture 2.2 (Jonsson-Mustaţă). *For any graded sequence of ideals $\mathfrak{a}_\bullet$, any valuation that calculates $\mathrm{lct}(\mathfrak{a}_\bullet)$ must be quasi-monomial.*

Recently, C. Xu made a significant progress on this conjecture and proved that minimizing valuations must be quasi-monomial. His work depends on our joint work ([16]) on approximations of minimizers and Birkar's work on the boundedness of complements. A related question is:

Question 2.1. *If for a graded sequence of ideals $\mathfrak{a}_\bullet$, v computes $\mathrm{lct}(\mathfrak{a}_\bullet)$, does v have a finitely generated associated graded ring?*

This is important because we have proved the following result which essentially reduces the conjecture 2.1 to the above question.

Theorem 2.3 ([17]). *For any Klt singularity, the minimizing quasi-monomial valuation with a finitely generated assocated graded ring is unique up to rescaling.*

2.1 Application: metric tangent cones at Klt singularities and equivariant K-stability

Let (X_i, ω_i) be a sequence of Kähler-Einstein Fano manifolds. By Gromov's compactness result, (X_i, ω_i) sub-sequentially converges to a limit metric space (X, d_∞). By the works of Donaldson-Sun and Tian, we know that X is homeomorphic to a normal $\mathbb{Q}$-Fano variety. A further question is:

what does the metric look like near the singularities of X? The first step to study this question is to understand the metric tangent cone C_xX for any point $x \in X$, which is defined to be a pointed Gromov-Hausdorff limit of a sequence of rescaled metric spaces centered at x. By CCT theory, we know that C_xX is a metric cone whose singular set has real Hausdorff codimension at least 4, and $(C_xX)^{\text{reg}}$ is Kähler-Ricci-flat. Donaldon-Sun further proved that C_xX is homeomorphic to an affine variety with a nontrivial torus action and is uniquely determined by the *metric* structure on the GH limit X, and they made the following

Conjecture 2.3 (Donaldson-Sun). *C_xX depends only on the algebraic structure of the germ (X, x).*

Donaldson-Sun essentially showed that there is a metrically defined K-semistable Fano cone associated to $x \in X$ that degenerates to the metric tangent cone C_xX which is K-polystable. So there are essentially two parts of the conjecture: uniqueness of the (metrically defined) K-semistable cone and the K-polystable cone. We proved in [17] that the K-semistable cone is associated to a quasi-monomial minimizer of the normalized volume functional. So the semistable part follows from Theorem 2.3. Moreover this shows that the algebraic quantity $\widehat{\text{vol}}(x, X)$ coincides with the volume density (in the sense of geometric measure theory) on Gromov-Hausdorff limits.

The polystable part of the conjecture 2.3 was further established by myself, together with Xiaowei Wang and Chenyang Xu. More precisely, we proved the following result which shows a very similar phenomenon as in the classical GIT theory:

Theorem 2.4 ([23]). *Each K-semistable Fano cone has a special degeneration to a K-polystable Fano cone, which is uniquely determined by the K-semistable Fano cone.*

The determination of metric tangent cones via algebraic structures can be used to study further regularity of singular Kähler-Einstein metrics (for example as shown in a work of Hein-Sun).

The method in the proof of Theorem 2.4 also allows to prove the following equivariant criterion for K-polystability:

Theorem 2.5 ([23]). *For any $\mathbb{Q}$-Fano variety with a torus action, the torus-equivariant K-polystability is equivalent to the usual K-polystability.*

This result allows to test effectively the K-polystability of T-varieties with low complexity. For smooth manifolds, this was proved by Datar-Székelyhidi using analytic methods based on partial C^0-estimates. An interesting problem is:

Problem 2.1. *Prove by purely algebraic methods the equivariant K-polystability criterion for general reductive Lie groups.*

2.2 Further applications/questions

Generalizing the work of Kento Fujita, Y. Liu observed that there is a local to global inequality between the volumes of K-semistable Fano variety and normalized volumes of singularities: if X is a K-semistable Fano variety, then for any point $x \in X$, the following inequality holds true:

$$(-K_X)^{\cdot n} \leq \frac{(n+1)^n}{n^n} \widehat{\text{vol}}(x, X). \tag{6}$$

This inequality has been used to study compactifications of the moduli space of smooth K-polystable Fano varieties by Spotti-Sun and Liu-Xu. A basic question in this direction is:

Conjecture 2.4. *$\widehat{\mathrm{vol}}(x, X) \leq 2(n-1)^n$ for any non-smooth n-dimensional klt singularity and the identity holds if and only if (X, x) is the ordinary double point.*

The above question is only answered affirmatively when $n = 2$ (by [14]), and when $n = 3$ by Liu-Xu who used classification results of 3-dimensional klt singularities. Since high dimensional classification is far from clear at present, new arguments are needed to study this questions and to get more effective bound for $\widehat{\mathrm{vol}}(x, X)$. There is also the following question:

Question 2.2. *How to bound the local fundamental groups of Klt singularities using their (normalized) volumes?*

I would like to mention another connection with other subject. Remark first that it is natural to extend the theory of normalized volume to the log pair case. For example there is a log version of (6) proved in my work with Y.Liu ([14]). Moreover in dimension 2 case, Borbon-Spotti observed that for special Klt pairs this invariant coincides with A. Langer's earlier invariant of local orbifold Euler numbers, and they conjectured that this is true in general. Combining the property of normalized volumes with a result on the slope stability of extension of logarithmic tangent sheaves (which generalizes an old result of Tian), I proved (in [20]) their expectation in the cone case and proposed a strategy to prove the general case in [24].

3 Algebraic study of K-stability and valuative criterions

As mentioned above, K-stability was introduced by G. Tian and S. Donaldson to study the existence of canonical Kähler metrics. It is a Hilbert-Mumford type criterion, which is defined in terms of test configurations and CM weights. A test configuration is a $\mathbb{C}^*$-equivariant degeneration of the polarized Fano variety $(X, -K_X)$. For each test configuration, one attaches a CM weight which generalizes the classical Futaki invariant associated to holomorphic vector fields. The name CM weight (also called Donaldson-Futaki invariant in the literature) comes from the fact that there is a functorial line bundle (called CM line bundle after Tian) on the base of the degeneration and the invariant is nothing but the weight of the $\mathbb{C}^*$-action on the central fibre of the CM line bundle. By now, there has been a large amount of works devoted to the algebraic study of K-stability.

In [5], I, together with C. Xu, discovered that K-stability of Fano varieties is "compatible" with the Minimal Model Program (MMP) in the sense that the CM weight is decreasing along the relative MMP. Starting with any test configuration, the MMP (with scaling) does various surgeries to "simplify" the total space and hence the central fibre of the degeneration, and ends with a nice one called special degeneration. Special degenerations were used by Tian to define K-stability because of a corresponding compactness result in metric geometry, while Donaldson used more general test configurations. As a consequence of the above monotonicity, we proved the equivalence of Tian and Donaldson's definition of K-stability for all Fano varieties.

Associated to any special test configuration of a $\mathbb{Q}$-Fano variety X, Boucksom-Hisamoto-Jonsson associated to a divisorial valuation ord_F on its function field $\mathbb{C}(X)$ (realized on some birational model $\mu : Y \to X$). In [12] I first realized that the CM weight associated to a special test configuration can be expressed in the following simple form:

$$\Theta_X(F) := A_X(\mathrm{ord}_F) - \frac{1}{(-K_X)^{\cdot n}} \int_0^{+\infty} \mathrm{vol}(\mu^*(-K_X) - x \cdot \mathrm{ord}_F) dx.$$

(K. Fujita obtained the same expression after my preprint) On the other hand, setting $(\mathfrak{C}, x) = C(X, -rK_X)$ for a divisible $r > 0$, ord_F induces a ray of valuations $\{v_t\}_{t \geq 0} \subset \mathrm{Val}_{\mathfrak{C},x}$ with $v_0 = \mathrm{ord}_X$.

The basic connection of $\Theta_X(F)$ with the normalized volume functional is given by the following formula proved in [12]:

$$\frac{d}{dt}\widehat{\mathrm{vol}}(v_t)\Big|_{t=0} = C\cdot\Theta_X(F). \tag{7}$$

where C is a constant depending only on X and r. This formula matches the calculation of Martelli-Sparks-Yau in their study of volume minimization property of Sasaki-Einstein metrics. Another important property is that $\widehat{\mathrm{vol}}(v_t)$ is convex in t. Combining above discussion, K-(semi)stability of Fano varieties can be now characterized in several ways:

Theorem 3.1 ([12, 14, 16]). *The following conditions are equivalent for any $\mathbb{Q}$-Fano variety X:*

(1) $(X, -K_X)$ is K-semistable;
(2) ord_X is a minimizer of $\widehat{\mathrm{vol}}_{\mathfrak{C},x}$;
(3) $\Theta_X(F)\geq 0$ for any divisorial valuation ord_F on $\mathbb{C}(X)$.

Moreover, if $\Theta_X(F)>0$ for any divisorial valuation ord_F then X is K-stable.

The converse to the last statement was written down by Blum-Xu based on an argument from [23].

As mentioned in section 1, in the process of proving equivariantly uniform version of Yau-Tian-Donaldson conjecture, I proved the following criterion for equivariantly uniform K-stability, which generalizes the valuative criterion for uniform K-stability of K. Fujita:

Theorem 3.2 ([26]). *Let $\mathbb{G}$ be a connected reductive subgroup of $\mathrm{Aut}(X)_0$. Then X is uniformly $\mathbb{G}$-stable if and only if there exists $\delta>1$ such that for any divisorial valuation $v := \mathrm{ord}_F$ over X, there exists an ξ in the Lie algebra of the center torus of $\mathbb{G}$ such that $A_X(v_\xi)\geq\delta\cdot S_{-K_X}(v_\xi)$.*

To prove this, I used the structure theory of T-varieties to twist the valuation v by the element ξ. Moreover besides using the above work with C. Xu on simplifying test configurations, I need to reformulate Hisamoto's notion of equivariantly uniform K-stability in terms of non-Archimedean metrics, and relate the twists of the different objects (test configurations, valuations and filtrations). In view of the results on our work on YTD conjecture and the development of the algebraic study of K-stability, a basic question to study is:

Problem 3.1. *For any $\mathbb{Q}$-Fano variety X, prove the equivalence of uniform K-stability with K-stability, and the equivalence of K-polystability with $\mathrm{Aut}(X)_0$-uniform K-stability.*

It is recently realized that this problem is also important in the algebraic approach to prove that moduli spaces of K-polystable $\mathbb{Q}$-varieties are compact. More importantly, based on the various criterions that were obtained, we want to study the following natural problem:

Problem 3.2. *Find effective ways to test the K-stability of Fano varieties.*

4 Other works and related developments

(1) In [12], we constructed proper moduli spaces of $\mathbb{Q}$-Gorenstein smoothable KE Fano varieties (there was also an independent work by Y. Odaka). To achieve this, we developed the family version of the conical continuity method and proved several technical results, including the separatedness of the moduli space, the Zariski openness of K-semistable ones and some stabilizer preserving property. The proof is a combination of analytic and algebraic tools, which depend on the work of CDS-Tian and Berman. Our theorem has been used in finding

explicit descriptions of moduli spaces of KE Fano varieties (e.g. cubic threefolds by Liu-Xu) Recently there are works by Blum, Liu and Xu etc. who aim to prove same results for all K-(semi)stable $\mathbb{Q}$-Fano varieties by using purely algebraic methods in the study of K-stability.

In a following work ([15]), we further proved quasi-projectivity of moduli spaces of smooth KE Fano manifolds. We achieved this by proving that the CM line bundle descends to become a nef and big line bundle over the proper moduli space. Moreover the augmented base locus of this line bundle is contained in the boundary set which parametrizes singular KE Fano varieties which are GH limits. The nef-big result was proved by showing that there is a canonically defined continuous Hermitian metric on the CM line bundle whose curvature current extends the canonical Weil-Petersen metric on the moduli space of smooth KE Fano manifolds. The continuity of this Hermitian metric is crucial and its proof uses the techniques that I developed in [11]. We still have the following

Problem 4.1. *Prove the projectivity of the proper moduli space.*

(2) Based on the work of CDS-Tian, earlier I proved that a smooth Fano manifold is K-semistable if and only if the Ding energy is bounded from below. I proved this result by showing that the Ding energy is continuous under special degenerations, which is essentially a result on the convergence of volume measures under degenerations. Later I generalized the latter result to general $\mathbb{Q}$-Fano varieties by using purely algebraic method discussed in section 3.

Theorem 4.1 ([11])**.** *If a $\mathbb{Q}$-Fano variety specially degenerates to a K-semistable Fano variety, then it is itself K-semistable.*

A related result that I proved even earlier is that a constant scalar curvature Kähler metric obtains the minimum of Mabuchi energy. This result was proved by using the quantization and approximation method based on the asymptotic expansion of Bergman kernels.

(3) Conical KE metrics were discussed in Tian's lecture notes in the 90's, who proposed to use it to derive Chern-number inequalities and bound the number of rational curves on algebraic surfaces. After the initial works of Jeffres and Jeffres-Mazzeo, the construction was however impeded by an unsatisfied linear theory. In 2012, S. Donaldson overcame the obstacle by establishing a Schauder estimate in the conical setting, and more importantly he proposed to use the conical KE metrics as a continuity method to attack the YTD conjecture. After this, the existence of conical KE metrics were still restricted to the case when cone angles are in the interval $(0, \pi]$ since otherwise the curvature of the reference metric is unbounded. In my joint appendix [9] with Y. Rubinstein, we first realized that the bisectional curvature of the reference metrics is actually bounded from above. This allows for the first time in Jeffres-Mazzeo-Rubinstein's work to construct conical KE metrics with angles bigger than π.

I have some other works on conical KE metrics. For example, I formulated the log-K-stability after Donaldson's work. Moreover, together with Song Sun, we proposed some interpolation-degeneration construction of conical KE metrics. The degeneration part, meaning to prove the boundedness of Ding energy of K-semistable Fano varieties, was refined in my work [11]. Some of this joint work was used in CDS-Tian's work. Moreover we applied our construction to answer affirmatively Gauntlett-Martelli-Sparks-Yau's question about the existence of Ricci-flat Kähler cone metrics (and K-semistability) on the 3-dim A_2 singularity ([6]). Since this contradicts Conti's work on classification of cohomogeneity one Sasaki-Einstein 5-manifolds, I also made calculations in [7] to point out the errors in his work.

(4) The classical continuity method for solving the Kähler-Einstein equation is given by:

$$Ric(\omega_t) = t\omega_t + (1-t)\tilde{\omega}. \tag{8}$$

Tian's early work shows that $\beta(X) := \sup\{t;\ \text{the above equation solvable}\}$ is in general less than 1. G. Székelyhidi proved that $\beta(X)$ does not depend on the reference $\tilde{\omega}$ and is indeed an invariant of X. In [3] I derived an explicit formula for this invariant on any toric manifold X in terms an invariant of the associated convex polytope. In [4] I further described the limit behavior of ω_t as $t \to \beta(X)$ for toric invariant metrics.

Later, Datar-Székelyhidi used the partial C^0-estimate method to obtain a description of $\beta(X)$ for smooth Fano manifold X in terms of twisted K-stability. More recently, $\beta(X)$ for general $\mathbb{Q}$-Fano varieties is studied using purely algebraic geometry. Following the works of Fujita-Odaka, Blum-Jonsson and Boucksom-Jonsson, a normalized version of $\Theta_X(F)$ can be used to define the stability threshold:

$$\delta(X) = \inf_{F:\text{divisorial}} \frac{A_X(F)(-K_X)^{\cdot n}}{\int_X \mathrm{vol}(-K_X - tF)dt}. \tag{9}$$

Cheltsov-Rubinstein-Zhang and Berman-Boucksom-Jonsson proved the identity $\delta(X) = \beta(X)$. There is a whole set of works for studying the minimization about δ-invariant by Blum, Jonsson, Y. Liu, C. Xu and others. These works parallel the minimization problem of normalized volumes. In some sense, these are local and global theories respectively. Some of these studies actually depend on the method in the study of normalized volume functionals.

(5) Back in 2014, motivated by Conlon-Hein's work which refines Tian-Yau's existence result of Asymptotically Conical (AC) Calabi-Yau metrics, I proved an analytic compactification result for AC Kähler metrics with regular tangent cones and determined optimal rate of convergence for Tian-Yau's AC Calabi-Yau metrics ([21]). The get optimal rate, besides the previous work of Conlon-Hein, I also used higher order deformation theory and related it to the linearity of holomorphic neighborhoods of complex submanifold as studied by Abate-Bracci-Tovena. The analytic compactification result can be seen as an analytic analogue of an algebraic result of Pinkham and was proved by adapting Newlander-Nirenberg's classical proof to the weighted setting. This work was later used by Conlon-Hein to give a classification of AC Calabi-Yau manifolds whose metric cones at infinitely have smooth links. Due to the technical nature of the arguments, this work is accepted only very recently by Duke Mathematical Journal after a few years' review.

References

[1] C. Li, Kahler-Einstein metrics and K-stability, Thesis (Ph.D.), Princeton University, 2012.

Published/Accepted Papers:

[2] C. Li, Constant scalar curvature Kähler metric obtains the minimum of K-energy, *Int. Math. Res. Not.*, 2011, no. 9, pp. 2161-2175.

[3] C. Li, Greatest lower bounds on Ricci curvature for toric Fano manifolds, *Adv. Math.*, **226**, 2011, no. 6, pp. 4921–4932.

[4] C. Li, On the limit behavior of metrics in the continuity method for the Kähler-Einstein problem on a toric Fano manifold, *Compos. Math.*, **148**, 2012, no. 6, pp. 1985–2003.

[5] C. Li and C. Xu, Special test configuration and K-stability of Fano varieties, *Ann. of Math.* **180**, 2014, no. 1, pp. 197–232.

[6] C. Li and S. Sun, Conical Kähler-Einstein metrics revisited, *Comm. Math. Phys.*, **331**, 2014, no. 3, pp. 927–973.

[7] C. Li, Numerical solutions of Kähler-Einstein metrics on $\mathbb{P}^2$ with conical singularities along a smooth quadric curve, *J. Geom. Anal.*, **25**, 2015, no. 3, pp. 1773–1797.

[8] C. Li, Remarks on logarithmic K-stability, *Commun. Contemp. Math.* **17**, 2015, no. 2, pp. 1450020.

[9] C. Li and Y.A. Rubinstein, Appendix A to "Kähler-Einstein metrics with singularities by T. Jeffres, R. Mazzeo and Y.A. Rubinstein", *Annals of Mathematics*, 183 (2016), pp 162-166.

[10] C. Li, A Pohožaev identity and critical exponents of some complex Hessian equations, *J. Partial Differ. Equ.*, **29**, 2016, no. 3, pp. 175–194.

[11] C. Li, Yau-Tian-Donaldson correspondence for K-semistable Fano manifolds, *J. Reine Angew. Math.*, **733**, 2017, pp. 55-85.

[12] C. Li, K-semistability is equivariant volume minimization, *Duke Math. J.*, **166**, 2017, no. 16, pp. 3147–3218.

[13] C. Li, Minimizing normalized volumes of valuations, *Math. Z.* **289**, 2018, no. 1-2, pp. 491–513.

[14] C. Li and Y. Liu, Kähler-Einstein metrics and volume minimization, *Adv. Math.*, **341** (2019), 440-492.

[15] C. Li, X. Wang and C. Xu, Quasi-projectivity of the moduli space of smooth Kähler-Einstein Fano manifolds, *Ann. Sci. Éc. Norm. Supér.*, (4) 51 (2018), no.3, 739-772.

[16] C. Li and C. Xu, Stability of valuations and Kollár components, accepted by *J. Eur. Math. Soc.*, arXiv:1604.05398.

[17] C. Li and C. Xu, Stability of valuations: higher rational rank, *Peking Math. J.* **1** (2018), no. 1, 1-79.

[18] C. Li, X. Wang and C. Xu, On the proper moduli spaces of smoothable Kähler-Einstein Fano varieties, *Duke Math. J.*, **168** (2019), no.8, 1387-1459.

[19] C. Li, G. Tian, Orbifold regularity of weak Kähler-Einstein metrics. *Advances in complex geometry*, 169-178, Contemp. Math., 735, Amer. Math. Soc., Providence, RI, 2019.

[20] C. Li, On the stability of extensions of tangent sheaves on Kahler-Einstein Fano/Calabi-Yau pairs, accepted by *Math. Ann.*, *arXiv:1803.01734*, 2018.

[21] C. Li, On sharp rates and analytic compactifications of asymptotically conical Kähler metrics, accepted by *Duke Math. J.*, arXiv:1405.2433.

Preprints:

[22] C. Li, G. Tian and F. Wang, On Yau-Tian-Donaldson conjecture for singular Fano varieties, arXiv.1711.09530.

[23] C. Li, X. Wang and C. Xu, Algebraicity of metric tangent cones and equivariant K-stability, arXiv:1805.03393.

[24] C. Li, Y. Liu. and C. Xu, A guided tour to normalized volumes, arXiv:1806.07112.

[25] C. Li, G. Tian and F. Wang, The uniform version of Yau-Tian-Donaldson conjecture for singular Fano varieties, arXiv:1903.01215.

[26] C. Li, On equivariantly uniform stability and Yau-Tian-Donaldson conjecture for singular Fano varieties, arXiv:1907.09399.