5.3. Difference equations and powers $A^k$.

- Fibonacci numbers: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

$$F_{k+2} = F_{k+1} + F_k \iff \begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$$

$$U_{k+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} U_k.$$

$$\Rightarrow U_k = A^k U_0.$$

If $A$ is diagonalizable, $A = S \Lambda S^{-1} \Rightarrow A^k = S \Lambda^k S^{-1}$.

$$\Rightarrow A^k U_0 = S \Lambda^k (S^{-1} U_0) = (\lambda_1, \ldots, \lambda_n) \begin{pmatrix} \lambda_1^k \\ \lambda_2^k \\ \vdots \\ \lambda_n^k \end{pmatrix} = c_1 \lambda_1^k + \ldots + c_n \lambda_n^k x_0.$$

reduce to find eigenvalues & eigenvectors of $A$ linear combination of $n$ "pure solutions" $\lambda_1 \ldots \lambda_n x_0.$

**Step 1:** $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1$

**Step 2:**
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

**Step 3:**
$$\lambda_1 = \frac{1 + \sqrt{5}}{2}; \quad \begin{pmatrix} 1 - \frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\sqrt{5}}{2} \\ 1 - \frac{\sqrt{5}}{2} \end{pmatrix} \Rightarrow V_1 = \begin{pmatrix} 1 \\ \frac{1 - \sqrt{5}}{2} \end{pmatrix}$$

$$\lambda_2 = \frac{1 - \sqrt{5}}{2}; \quad \begin{pmatrix} 1 - \frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 - \frac{\sqrt{5}}{2} \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} \frac{1 - \sqrt{5}}{2} \\ 1 \end{pmatrix}$$

$$\Rightarrow S = \begin{pmatrix} \frac{1 + \sqrt{5}}{2} & \frac{1 - \sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}.$$
\[ A^k = S \Lambda^k S^{-1} = \begin{pmatrix} \left(\frac{1 + \sqrt{5}}{2}\right)^k & \left(\frac{1 - \sqrt{5}}{2}\right)^k \\ 0 & \left(\frac{1 + \sqrt{5}}{2}\right)^k \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\frac{1 + \sqrt{5}}{2} \\ 1 & \frac{1 - \sqrt{5}}{2} \end{pmatrix} \]

\[ = \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1 + \sqrt{5}}{2}\right)^k + \left(\frac{1 - \sqrt{5}}{2}\right)^k & \left(\frac{1 + \sqrt{5}}{2}\right)^k - \left(\frac{1 - \sqrt{5}}{2}\right)^k \\ \left(\frac{1 + \sqrt{5}}{2}\right)^k - \left(\frac{1 - \sqrt{5}}{2}\right)^k & \left(\frac{1 + \sqrt{5}}{2}\right)^k + \left(\frac{1 - \sqrt{5}}{2}\right)^k \end{pmatrix} \frac{1 + \sqrt{5}}{2} \frac{1 - \sqrt{5}}{2} \]

\[ = \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1 + \sqrt{5}}{2}\right)^k + \left(\frac{1 - \sqrt{5}}{2}\right)^k & 2 \left(\frac{1 + \sqrt{5}}{2}\right)^k - \left(\frac{1 - \sqrt{5}}{2}\right)^k \\ 2 \left(\frac{1 + \sqrt{5}}{2}\right)^k - \left(\frac{1 - \sqrt{5}}{2}\right)^k & \left(\frac{1 + \sqrt{5}}{2}\right)^k + \left(\frac{1 - \sqrt{5}}{2}\right)^k \end{pmatrix} \frac{1 + \sqrt{5}}{2} \frac{1 - \sqrt{5}}{2} \]

Ex: \( U_0 = (1, 0) \Rightarrow U_k = A^k U_0 = \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1 + \sqrt{5}}{2}\right)^k - \left(\frac{1 - \sqrt{5}}{2}\right)^k \\ \left(\frac{1 + \sqrt{5}}{2}\right)^k - \left(\frac{1 - \sqrt{5}}{2}\right)^k \end{pmatrix} \]

\( \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} \)

\[ \Rightarrow F_k = \frac{1}{\sqrt{5}} \left[ \left(\frac{1 + \sqrt{5}}{2}\right)^k - \left(\frac{1 - \sqrt{5}}{2}\right)^k \right] \text{ is an integer!} \]

\[ \left| \frac{1 - \sqrt{5}}{2} \right|^k < \frac{1}{1.236} \left(\frac{1 - \sqrt{5}}{2}\right)^k \leq \frac{1}{2} \]

\[ \Rightarrow F_k \text{ is the nearest integer to } \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^k \]

\[ \frac{F_{k+1}}{F_k} \rightarrow \lambda = \frac{1 + \sqrt{5}}{2} \]
Ex: each year \( \frac{1}{10} \) of the people outside California move in, \( \frac{3}{10} \) of the people inside California move out.

\[ Y_k; \text{number outside}, \quad Z_k; \text{number inside}; \]
\[ \begin{pmatrix} Y_k+1 \\ Z_k \end{pmatrix} = \begin{pmatrix} \frac{9}{10} & \frac{2}{10} \\ \frac{8}{10} & \frac{8}{10} \end{pmatrix} \begin{pmatrix} Y_k \\ Z_k \end{pmatrix} = A \begin{pmatrix} Y_k \\ Z_k \end{pmatrix} \]

Two essential properties of a Markov matrix \( A \):

1. Each column of \( A \) adds up to 1.
2. The matrix has no negative entries.

To find \( A^k \), diagonalize \( A \) by finding eigenvalues \& eigenvectors:

1. 
\[ |A-\lambda I| = \begin{vmatrix} \frac{9}{10} - \lambda & \frac{2}{10} \\ \frac{8}{10} & \frac{8}{10} - \lambda \end{vmatrix} = \lambda^2 - 1.7\lambda + 0.72 - 0.2 = \lambda^2 - 1.7\lambda + 0.7 \]
2. 
\[ \lambda^2 - 1.7\lambda + 0.7 = (\lambda - 1)(\lambda - 0.7) \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 0.7 \]
3. 
\[ \lambda_1 = \begin{pmatrix} 0.9 - 1 & 0.2 \\ 1 & 0.8 - 1 \end{pmatrix} = \begin{pmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} \] \( \Rightarrow \nu_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \)
\[ \lambda_2 = 0.7 \begin{pmatrix} 0.9 - 0.7 & 0.2 \\ 0.1 & 0.8 - 0.7 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \] \( \Rightarrow \nu_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)

\[ S = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow A^k = S \Lambda_k^k S^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.7^k \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} \]
\[ = \frac{1}{3} \begin{pmatrix} 2 & 0.7^k \\ 1 & -0.7^k \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 + 0.7^k & 2 - 2 \cdot 0.7^k \\ 1 - 0.7^k & 1 + 2 \cdot 0.7^k \end{pmatrix} \]

\[ \Rightarrow \begin{pmatrix} Y_k \\ Z_k \end{pmatrix} = A^k \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \]
\[
\begin{pmatrix}
\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} \\
\begin{bmatrix} y_2 \\ z_2 \end{bmatrix}
\end{bmatrix} = \begin{pmatrix} \frac{2}{3} & (0.7)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\begin{bmatrix} -0.7^k \\ 1 \end{bmatrix} \end{pmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \left( y_0 + z_0 \right) \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} + \left( y_0 - 2z_0 \right) \begin{pmatrix} 0.7^k \\ 1 \end{pmatrix}
\]

\[
\begin{pmatrix} y_0 + z_0 \\ -\frac{1}{3} \end{pmatrix} \xrightarrow[k \to \infty]{} c_1 x_1 \quad \Leftrightarrow \quad c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2
\]

\[
\lambda_1 = 1 \quad \lambda_2 = 0.7
\]

\[
\begin{pmatrix} y_x \\ z_x \end{pmatrix} \leftarrow \text{steady state.}
\]

**Theory of Markov process:**

Then, A Markov matrix \( A \) has all \( a_{ij} \geq 0 \), with each column adding to 1

(a) \( \lambda_1 = 1 \) is an eigenvalue of \( A \)

(b) The eigenvector associated to \( \lambda_1 = 1 \) is nonnegative. (steady state:)

\( A x_0 = x_0 \)

(c) The other eigenvalues satisfies \( |\lambda_1| \leq 1 \).

(d) If \( A \) or any power of \( A \) has all positive entries, then other \( \lambda_0 \)

are below 1. \( AKU_0 \xrightarrow[k \to \infty]{} (a \text{ multiple of } x_1) \) which is the

\( U_0 \leftarrow \text{steady state.} \)

**Proof:**

(a) each column adds to 1 \( \Rightarrow \) \( A-I \) has linearly dependent rows

(rows of \( A-I \) add to 0 vector)

\( \Rightarrow \) \( A-I \) is singular \( \Rightarrow \) \( \det(A-I) = 0 \) \( \Rightarrow \) 1 is an eigenvalue

(b) -(d) are consequences of Perron-Frobenius theorem.