Notes on Kähler-Ricci flow

immediate

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1 Kähler metrics on complex manifolds

Definition 1. Almost complex structure: $J : T_R M \rightarrow T_R M$ satisfies $J^2 = -Id$.

Definition 2. (complex manifolds) An holomorphic atlas: $\{(U_\alpha, z_\alpha = \{z_i^\alpha\} : U_\alpha \rightarrow \mathbb{C}^n)\}$ such that the transition function $z_\alpha \circ z_\beta^{-1} : U_\alpha \cap U_\beta \rightarrow \mathbb{C}^n$ is holomorphic.

If $M$ is a complex manifold, then there is a natural almost complex structure on $T_R M$ induced by the standard complex structure on $\mathbb{C}^n \cong \mathbb{R}^n$:

$$J \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}, \quad J \frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}.$$
The complexified tangent bundle $T_C M = T_C M \otimes \mathbb{C}$ splits into $\pm i$-eigenspace of $J$:

$$T^{(1,0)} M = \text{span}_\mathbb{C} \left\{ \frac{\partial}{\partial z^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} - i \frac{\partial}{\partial y^i} \right) \right\}, \quad T^{(0,1)} M = \overline{T^{(1,0)} M}.$$  

$J$ is integrable in the sense that

$$[T^{(1,0)}, T^{(1,0)}] \subseteq T^{(1,0)} M.$$  

Reversely we have the Newlander-Nirenberg theorem:

**Theorem 1** (Newlander-Nirenberg). If $J$ is a $C^{2n+\alpha}$ almost complex structure on $M$ that is integrable, then there exists holomorphic atlas that makes $M$ a complex manifold.

**Remark 1.** This should be compared with Frobenius theorem for integrable distributions and can be seen as a generalization of the isothermal coordinate theorem in real dimensional 2 to higher dimensions.

From now on, assume $(M, J)$ is a complex manifold. $g$ is a Riemannian metric that is compatible with $J$ in the sense that: $g(JX, JY) = g(X, Y)$. In this case, we can define a 2-form by:

$$\omega(X, Y) = g(JX, Y),$$

In this case we say that $g$ is an Hermitian metric and call $\omega_g$ to be the Kähler form associated to $g$. Notice that $g(T^{(1,0)} M, T^{(1,0)} M) = 0$. So locally under holomorphic chart $(U, \{z^i\})$, we can write:

$$g = \sum_{i,j} g_{ij} \left( dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i \right). \quad (1)$$

where $g_{ij}$ forms a positive definite Hermitian matrix: $\overline{g_{ij}} = g_{ji}$ and $g_{ij} \xi^i \xi^j > 0$ if $\xi = (\xi^1, \ldots, \xi^n) \neq 0$. The associated Kähler form is a (1,1) form that has local expression as:

$$\omega = \sqrt{-1} \sum_{i,j} g_{ij} dz^i \wedge d\bar{z}^j. \quad (2)$$

**Definition 3.** A Riemannian metric $g$ on a complex manifold $(M, J)$ is called a Kähler metric if $g$ is an Hermitian metric and $d\omega_g = 0$.

**Proposition 1.** Assume $g$ is an Hermitian metric on $(M, J)$. Then the following condition is equivalent:


1. $\nabla^{\mathcal{LC}} J = 0$, or equivalently $\nabla^{\mathcal{LC}} \omega_g = 0$.

2. $d\omega_g = 0$, equivalently under local holomorphic coordinate chart we have

$$\frac{\partial g_{ij}}{\partial z^k} = \frac{\partial g_{kj}}{\partial z^i}, \forall i, j, k. \quad (3)$$

3. At any point $p \in M$, there exists a holomorphic chart $(U, \{z^i\})$ centered at $p$ (i.e. $z^i(p) = 0$) such that $g_{ij} = \delta_{ij} + O(|z|^2)$.

Proof. Because $d = e^i \wedge \nabla^L_{\omega_i}$, 1 implies 2 is clear. To see 2 implies 1, we first derive the following basic consequence of $d\omega_g = 0$ or equivalently of (3). Under holomorphic coordinate chart $\{z^i\}$, the following components of the Levi-Civita connection form vanish:

$$\Gamma^s_{ij} = 0, \text{ or } \bar{k}; \Gamma^k_{ij} = 0 = \Gamma^k_{ij}, \forall i, j, k.$$ In other words, we have:

$$\nabla_{\partial_{z^i}} \partial_{z^j} = 0, \quad \nabla_{\partial_{z^i}} \partial_{\bar{z}^j} = 0, \quad \forall i, j. \quad (4)$$

and

$$\nabla_{\partial_{\bar{z}^i}} \partial_{z^j} \in T^{(1,0)} M, \text{ and } \nabla_{\partial_{\bar{z}^i}} \partial_{\bar{z}^j} \in T^{(0,1)} M.$$ From the above equality, we see that $T^{(1,0)}$ and $T^{(0,1)}$ are preserved under parallel transport. Now we can calculate to see that $\nabla J = 0$.

$$(\nabla_{\partial_i} J)(\partial_j) = \nabla_{\partial_i} (J \partial_j) - J(\nabla_{\partial_i} \partial_j) = \sqrt{-1} \Gamma^k_{ij} \partial_k - J(\Gamma^k_{ij} \partial_k) = 0,$$

and $$(\nabla_{\partial_i} J)(\partial_j) = 0.$$ 3 implies 2 is clear. Assume 2 and write:

$$\omega_g = \sqrt{-1} g_{ij} dz^i \wedge \bar{z}^j \text{ with } g_{ij} = \delta_{ij} + a_{ijk} z^k + a_{ijk} \bar{z}^k + O(|z|^2).$$

Because $g_{ar{i}j} = g_{ar{i}j}$, we have $\overline{a_{ijk}} = a_{ar{i}jk}$. Consider the coordinate change $z^i = w^i + \frac{1}{2} c^j_{ik} w^j w^k + O(|w|^2)$. Then we have:

$$\omega_g = \sqrt{-1} \left( \delta_{ij} + a_{ijk} w^k + a_{ijk} \bar{w}^k \right) \left( dw^i + c^p_{ij} w^p dw^q \right) \wedge \left( d\bar{w}^j + c^q_{ij} \bar{w}^r d\bar{w}^s \right)$$

$$= \sqrt{-1} \left( dw^i \wedge d\bar{w}^j + (a_{ij} + c^j_{ik}) w^k dw^i \wedge d\bar{w}^j + (a_{ij} + c^j_{ik}) \bar{w}^k dw^i \wedge d\bar{w}^j \right).$$

So we need to choose $c^j_{ki} = a_{ijk}$, so that $\overline{c^j_{ki}} = a_{ar{i}jk}$, and $c^j_{ij} = a_{ar{i}jk}$. On the other hand, we have $a_{ijk} = c^j_{ki} = c^j_{ik} = a_{kji}$ which is guaranteed exactly because $a_{ijk} = \partial_k g_{ij}|_{z=0} = \partial_z g_{kj}|_{z=0} = a_{kji}$. \qed
The curvature tensor of a Kähler metric is easier to compute using the complex coordinates.

\[ \nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k, \text{ with } \Gamma^k_{ij} = g^{kl} \partial_l g_{ij}. \]

\[ \nabla_{\partial_i} \partial_j = 0 = \nabla_{\partial_i} \partial_j. \]

So the curvature is (use \( \partial_i = \partial_z i \) and \( \partial_j = \partial_{\bar{z}} j \))

\[ R(\partial_z i, \partial_{\bar{z}} j) \partial_k = R_{ij}^k = -\partial_j g^{kl} \partial_i g_{kl}. \]

\[ \text{So we get:} \]

\[ R_{ij} = R_{ji} = R_{kji} = -\partial_j \left( g^{kl} \partial_i g_{kl} \right) = -\partial_j \partial_i \log \det(g_{kl}). \]

The Ricci curvature

\[ R_{ij} = \text{Ric}(\omega) = -\sqrt{-1} \bar{\partial} \partial \log \omega^n = -\sqrt{-1} \sum_{i,j} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det(g_{kl}) dz^i \wedge d\bar{z}^j. \]

We can give the Chern-Weil explanation of this formula. Any volume form \( \Omega \) induces an Hermitian metric on \( K^{-1} \) by

\[ |\partial z_1 \wedge \cdots \wedge \partial z_n|_{\Omega}^2 = \frac{2^n \Omega}{\sqrt{-1} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n}. \]

By abuse of notation, we denote the Chern curvature of the \( (K^{-1}, |\cdot|_{\Omega}) \) to be \( \text{Ric}(\Omega) \) then we see that \( \text{Ric}(\omega) = \text{Ric}(\omega^n) \). So we see that the Ricci form is a closed (1,1) form representing the class \( 2\pi c_1(M) \).

\[ S(\omega_\phi) = g_\phi^{ij} \text{Ric}(\omega_\phi)_{ij}, \quad S = \frac{\langle n(2\pi c_1(M)) [\omega]^{n-1}, [M] \rangle}{\langle [\omega]^n, [M] \rangle} \]

are the scalar curvature of \( \omega_\phi \) and average of scalar curvature. Note that \( S \) is a topological constant.

**Lemma 1 (\( \partial \bar{\partial} \)-Lemma).** If \( [\omega_1] = [\omega_2] \), then there exists \( \phi \in C^\infty(M) \) such that \( \omega_2 - \omega_1 = \sqrt{-1} \partial \bar{\partial} \phi. \)
Proof. \( \omega_2 - \omega_1 = d\eta = \partial_\eta^{(1,0)} + \partial_\eta^{(0,1)} \). Consider the Hodge decomposition of \( \eta^{(1,0)} \) with respect to the operator \( \partial \), we get \( \eta^{(1,0)} = \partial \alpha + \partial^\ast \beta + \gamma \) where \( \gamma \in \text{Ker}(\square_\partial) = \text{Ker}(\square_\bar{\partial}) \). So \( \partial \eta^{(1,0)} = \partial \partial \alpha + \partial \partial^\ast \beta \). On the other hand, because \( 0 = \partial (\omega_2 - \omega_1) = \partial \partial \eta^{(1,0)} = \partial \partial \partial^\ast \beta \) and \( \partial \partial^\ast = -\partial^\ast \partial \), we have

\[
0 = \langle \partial \partial \partial^\ast \beta, \partial \partial^\ast \beta \rangle = \langle \partial \partial^\ast \beta, \partial^\ast \partial \partial^\ast \beta \rangle = -\| \partial \partial^\ast \beta \|_{L^2}^2.
\]

So \( \partial \partial^\ast \beta = 0 \). The term \( \partial_\eta^{(0,1)} \) is dealt with similarly. \( \square \)

**Exercise 1.** Use the \( \partial \)-Poincaré lemma and holomorphic Poincaré lemma to prove the local \( \partial \bar{\partial} \)-lemma on polydisks.

This Lemma is very useful because it reduces equations on Kähler metrics to equations involving Kähler potentials.

**Definition 4.** Fix a reference metric \( \omega \) and define the space of smooth Kähler potentials as

\[
\mathcal{H} := \mathcal{H}_\omega = \{ \phi \in C^\infty(M) | \omega_\phi := \omega + \sqrt{-1} \bar{\partial} \partial \phi > 0 \} \quad (5)
\]

**Remark 2.** The set \( \mathcal{H} \) depends on reference Kähler metric \( \omega \). However in the following, we will omit writing down this dependence, because it’s clear that \( \mathcal{H} \) is also the set of Hermitian metrics \( h \) on \( L \) whose curvature form

\[
\omega_h := -\sqrt{-1} \bar{\partial} \partial \log h
\]

is a positive \((1,1)\) form on \( X \). Since \( \omega_\phi \) determines \( \phi \) up to the addition of a constant, \( \mathcal{H}/\mathbb{C} \) is the space of smooth Kähler metric in the Kähler class [\( \omega \)]. By abuse of language, sometimes we will not distinguish \( \mathcal{H} \) and \( \mathcal{H}/\mathbb{C} \).

**Calabi problem:** Consider the Ricci curvature mapping:

\[
[\bar{\omega}] \longrightarrow 2\pi c_1(M) \\
\omega \mapsto \text{Ric}(\omega)
\]

**Question:** Subjectivity (Yau’s theorem on prescribing Ricci curvature) and injectivity (by maximum principle). Transform to the PDE for the potential function: for any \( \eta \in 2\pi c_1(M) \) and \( \omega = \bar{\omega} + \sqrt{-1} \bar{\partial} \partial \varphi \), we have:

\[
\text{Ric}(\omega) - \eta = \text{Ric}(\omega) - \text{Ric}(\bar{\omega}) + \text{Ric}(\bar{\omega}) - \eta \\
= -\sqrt{-1} \bar{\partial} \partial \log \frac{\omega^n}{\bar{\omega}^n} + \sqrt{-1} \bar{\partial} \partial f.
\]
So the equation $Ric(\omega) = \eta$ is equivalent to the equation involving only the potential function $\varphi$:

$$(\bar{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi) = e^{f - c}\bar{\omega}.$$ 

This is a complex Monge-Ampère type equation, since in local holomorphic coordinates, we have:

$$\det \left( \hat{g}_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) = e^{f - c} \det (g_{ij}).$$

Basic examples: Kähler manifolds with constant holomorphic sectional curvatures:

$$R_{ijkl} = \mu (g_{ij}g_{k\bar{l}} + g_{i\bar{l}}g_{j\bar{k}}), \quad \mu = -1, 0, 1.$$ 

Notation: $B^n = \{ z \in \mathbb{C}^n; |z| < 1 \}$,

$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^* = \mathbb{C}^n \cup \mathbb{P}^{n-1}.$

| $\mathbb{B}^n$ | $\omega_{\mathbb{B}} = -\sqrt{-1} \partial \bar{\partial} \log (1 - |z|^2)$ | $\mathbb{B}^n/\Gamma, \Gamma < \text{PSU}(n,1)$ |
| $\mathbb{C}^n$ | $\omega_{\mathbb{C}} = \sqrt{-1} \partial \bar{\partial} |z|^2$ | $\mathbb{C}^n/\Lambda, \Lambda \cong \mathbb{Z}^{2n}$ |
| $\mathbb{P}^n$ | $\omega_{\mathbb{FS}} = \sqrt{-1} \partial \bar{\partial} \log (1 + |z|^2)$ | $\mathbb{P}^n$ |

2 Kähler-Ricci flow: maximal existence time

$$\frac{\partial g}{\partial t} = -Ric(g) \iff \frac{\partial \omega}{\partial t} = -Ric(\omega).$$ 

(6)

The Ricci flow preserves the Kähler condition of the metric:

$$\frac{\partial}{\partial t} d\omega = d \frac{\partial \omega}{\partial t} = -d \left( Ric(\omega) \right) = 0.$$ 

$$\frac{\partial \omega}{\partial t} = -Ric(\omega), \quad \omega(0) = \omega_0.$$ 

(7)

Evolution of cohomology class:

$$\frac{d}{dt} [\omega_t] = -2\pi c_1(X) = 2\pi c_1(K_X) \Rightarrow [\omega_t] = [\omega_0] + 2\pi tc_1(K_X).$$

From now on, we fix the initial Kähler class $\alpha = [\omega_0] \in H^{1,1}(M, \mathbb{R})$. The Kähler class is positive definite if and only if $t \in [0, T_{\text{max}})$ where

$$T_{\text{max}} := T_{\text{max}}(\alpha) = \sup \{ t; \alpha + t2\pi c_1(K_M) > 0 \}.$$
Theorem 2 (Tian-Zhang). Fix a Kähler class $\alpha > 0 \in H^{1,1}(M, \mathbb{R})$. For any Kähler metric $\omega_0 \in \alpha$, the Kähler-Ricci flow exists if and only if $t \in [0, T_{\text{max}})$.

Example 1. Let $M = \mathbb{CP}^2 \# \mathbb{CP}^2 = \text{Bl}_p \mathbb{CP}^2$. Let $\pi : M \to \mathbb{CP}^2$ be the blow down map. Denote by $H$ the pull back of the hyperplane class of $\mathbb{CP}^2$. The Kähler cone is given by:

$$KC = \{ aH - bE; a > 0, b > 0, a > b \}.$$ 

$c_1(K_X) = -3H + E$. Fix a Kähler class $[\omega_0] = aH - bE \in KC$. So $[\omega_t] = (a - 3t)H - (b - t)E$. There are three cases:

1. $b < 3a$. $T_{\text{max}} = b$. $M \to \mathbb{CP}^2$.
2. $b = 3a$. $T_{\text{max}} = b = 3a$. $M \to \text{pt}$.
3. $b > 3a$. $T_{\text{max}} = \frac{a - b}{2}$. $M \to \mathbb{P}^1$.

Proof. Fix any $T \in [0, T_{\text{max}})$, there exists $\epsilon > 0$ such that because $[\omega_0] + (T + \epsilon)2\pi c_1(K_M) > 0$, there exists a Kähler metric $\hat{\omega}_{T+\epsilon} \in [\omega_0] + (T + \epsilon)2\pi c_1(K_M)$ so that $\hat{\omega}_{T+\epsilon} - \omega_0 \in (T + \epsilon)2\pi c_1(K_M)$. By the $\partial \bar{\partial}$-lemma, there exists a smooth volume form $\Omega = \Omega_{T+\epsilon}$ such that

$$\psi := \text{Ric}(\Omega) = -\sqrt{-1}\partial \bar{\partial} \log \Omega = -\frac{1}{T+\epsilon}(\hat{\omega}_{T+\epsilon} - \omega_0)/(T + \epsilon).$$

Define the changing reference metric

$$\hat{\omega}_t = \omega_0 + t(-\text{Ric}(\Omega)) = \left(1 - \frac{t}{T + \epsilon}\right)\omega_0 + \frac{t}{T + \epsilon}\hat{\omega}_{T+\epsilon} \in [\omega_0] + t2\pi c_1(K_M).$$

Then by the $\partial \bar{\partial}$-lemma, we can write $\omega_t = \hat{\omega}_t + \sqrt{-1}\partial \bar{\partial} \varphi$ with $\varphi \in C^\infty(M)$. Taking derivative with respect to $t$ on both sides, we get:

$$\frac{\partial \omega_t}{\partial t} = \frac{\partial \hat{\omega}_t}{\partial t} + \sqrt{-1}\partial \bar{\partial} \frac{\partial \varphi}{\partial t} = -\text{Ric}(\Omega) + \sqrt{-1}\partial \bar{\partial} \frac{\partial \varphi}{\partial t}.$$ 

On the other hand, we have:

$$-\text{Ric}(\omega) = -\text{Ric}(\omega) + \text{Ric}(\Omega) - \text{Ric}(\Omega) = \sqrt{-1}\partial \bar{\partial} \log \frac{\omega^n}{\Omega} - \text{Ric}(\Omega).$$

So the Kähler-Ricci flow is equivalent to the flow:

$$\sqrt{-1}\partial \bar{\partial} \frac{\partial \varphi}{\partial t} = \sqrt{-1}\partial \bar{\partial} \log \frac{\omega^n}{\Omega}.$$
So up to a constant we get the following flow equation:

\[
\frac{\partial \varphi}{\partial t} = \log \left( \frac{\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi}{\Omega} \right)^n, \quad \varphi(0) = 0. \tag{8}
\]

Notice that this is equivalent to the Monge-Ampère type equation:

\[
(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{\frac{\partial \varphi}{\partial t}} \Omega, \quad \varphi(0) = 0.
\]

We want to show that this flow has a solution for \( t \in [0, T + \epsilon) \). There are several steps:

- \((C^0\)-estimate of \( \varphi \) and \( \partial \varphi/\partial t \))

Assume \( \min_{x \in X} \varphi(x, t) = \varphi(x_t) \) for \( x_t \in X \). Then at \( x_t \) we have:

\[
\frac{\partial \varphi}{\partial t}(x_t) \geq \log \frac{\hat{\omega}_t^n}{\Omega}(x_t) \geq n \log \epsilon + A. \tag{9}
\]

The last inequality is because:

\[
\hat{\omega}_t = \left(1 - \frac{t}{T + \epsilon}\right) \omega_0 + \frac{t}{T + \epsilon} \hat{\omega}_{T+\epsilon} \geq \frac{\epsilon}{T + \epsilon} \omega_0.
\]

By the minimum principle, we get \( \min \varphi \) is non-decreasing so that

\[
\varphi(x, t) \geq (n \log \epsilon + A)t. \tag{10}
\]

For the upper bound, we get:

\[
\frac{\partial u_{\max}}{\partial t} \leq \log \frac{\hat{\omega}_t^n}{\Omega} \leq \max_{x \in X} \log \frac{(\omega_0 + \hat{\omega}_{T+\epsilon})^n}{\Omega} = B.
\]

The last inequality is because \( \hat{\omega}_t \leq \omega_0 + \hat{\omega}_{T+\epsilon} \). This implies that:

\[
u \leq u_{\max} \leq Bt. \text{ for } t \in [0, T].
\]

Next we consider the bound for \( \partial u/\partial t \). Taking derivative of (8), we get:

\[
\frac{\partial}{\partial t} \varphi = n(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^{n-1} \wedge (-Ric(\Omega) + \sqrt{-1} \partial \bar{\partial} \varphi_t) = -tr_{\omega_t}(Ric(\Omega)) + \Delta_t \varphi.
\]

Using the equality \( \hat{\omega}_t = \omega_0 - tRic(\Omega) \), it follows:

\[
\left( \frac{\partial}{\partial t} - \Delta_t \right)(t \varphi - \varphi) = t \partial_t \varphi + t(-tr_{\omega_t}(Ric(\Omega)) + \Delta_t \varphi)
\]

\[
= \Delta_t(t \varphi - \varphi) + tr_{\omega_t}(\omega_t - \hat{\omega}_t) - t \cdot tr_{\omega_t}(Ric(\Omega))
\]

\[
= \Delta_t(t \varphi - \varphi) + n - tr_{\omega_t} \omega_0.
\]
We rewrite it as:

$$\frac{\partial}{\partial t} (t\dot{\varphi} - \varphi - nt) = \Delta(t\dot{\varphi} - \varphi - nt) - tr_{\omega_t}(\omega_0).$$  \hspace{1cm} (11)$$

Because $tr_{\omega_t}\omega_0 > 0$, by maximal principle we know that the maximum $t\dot{\varphi} - u - nt$ is nondecreasing, so we get:

$$t\frac{\partial \varphi}{\partial t} - \varphi - nt \leq 0.$$ 

By combining the local existence for small time and the uniform upper bound of $\varphi$, we get:

$$\frac{\partial \varphi}{\partial t} \leq n + t^{-1} \sup_{M} \varphi \leq n + C.$$ 

To get lower bound of $\frac{\partial \varphi}{\partial t}$, we calculates:

$$\frac{\partial}{\partial t} ((T + \epsilon - t)\dot{\varphi} + \varphi) = (T + \epsilon - t)\frac{\partial}{\partial t} \dot{\varphi} = (T + \epsilon - t)(-tr_{\omega_t}\psi + \Delta_t\dot{\varphi})$$

$$= \Delta_t((T + \epsilon - t)\dot{\varphi} + \varphi) - tr_{\omega_t}(\omega_t - \hat{\omega}_t) - (T + \epsilon - t)tr_{\omega_t}\psi$$

$$= \Delta_t((T + \epsilon - t)\dot{\varphi} + \varphi) - n + tr_{\omega_t}(\omega_0 - (T + \epsilon)\psi).$$

We rewrite it as:

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)((T + \epsilon - t)\dot{\varphi} + \varphi + nt) = tr_{\omega_t}(\omega_0 - (T + \epsilon)\psi).$$ \hspace{1cm} (12)$$

Because $tr_{\omega_t}(\omega_0 - (T + \epsilon)\psi) = tr_{\omega_t}\hat{\omega}_{T+\epsilon} > 0$, so by the maximal principle, we get:

$$(T + \epsilon - t)\frac{\partial \varphi}{\partial t} + \varphi + nt \geq (T + \epsilon)\min_{t=0} \frac{\partial \varphi}{\partial t} > -C.$$ 

**Exercise 2.**

$$\frac{\partial \dot{\varphi}}{\partial t} = -R(t)$$

where $R(t)$ is the scalar curvature of $\omega_t$. So the bound of $\partial \dot{\varphi}/\partial t$ is equivalent to the bound on the scalar curvature of $\omega_t$. There is always a uniform lower bound of $R(t)$ for general Ricci flow. So there is a uniform upper bound of $\dot{\varphi} = \partial \dot{\varphi}/\partial t$. We also have:

$$\Delta \dot{\varphi} = -R + tr_{\omega_t}\psi.$$
1. There is a uniform upper bound of $\tilde{\phi}$ depending only on $T_{\max}$. One can actually directly use Exercise 2 to get this.

2. Integrate twice the upper bound of $\tilde{\phi}$ to get upper bound of $\phi$.

3. Let $\tilde{\omega}_t = \left(1 - \frac{t}{T_{\max}}\right) \omega_0 + \frac{t}{T_{\max}} \tilde{\omega}_{T_{\max}}$. Then $\tilde{\omega}_t$ is still smooth Kähler metric for $t \in [0, T_{\max})$. Previous calculation shows that:

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) ((T_{\max} - t)\tilde{\phi} + \phi + nt) = \Delta_t((T_{\max} - t)\tilde{\phi} + \phi) - n + tr_\omega \tilde{\omega}_{T_{\max}}.$$

So by the maximal principle, we get:

$$(T_{\max} - t)\frac{\partial \phi}{\partial t} + \phi + nt \geq T_{\max} \min_{t=0} \frac{\partial \phi}{\partial t} \geq -C.$$

By the upper bound of $\partial u/\partial t$, we get uniform lower bound of $\phi$.

- $(C^2$-estimate of $u$)

**Lemma 2.** For any family of Kähler metrics $\tilde{\omega}_t$:

$$(\partial_t - \Delta_t) tr_\omega \tilde{\omega}_t = -tr_\omega (\partial \tilde{\omega}_t) + \tilde{R}_{ijk} g^{ij} g^{k\ell} - g^{ij} g^{k\ell} \tilde{g}_{\ell ij} \tilde{g}_{ji} \tilde{g}_{ij}. \quad (13)$$

**Proof.** Choose local coordinate $\{z^i\}$ so that $\partial_k g_{ij}(x) = 0$ for any $i, j, k$. Then we an calculate:

$$\Delta(g^{ij} \tilde{g}_{ij}) = (g^{ij} \tilde{g}_{ij})_{kl} g^{k\ell} = \left(-g^{ij} g^{pq} g^{\ell q} \tilde{g}_{ij} + g^{ij} g_{\ell ij}\right) g^{k\ell}$$

$$= \left(-g^{ij}(\partial_k \partial_{pq}) g^{\ell q} \tilde{g}_{ij} + g^{ij} g \partial_k \partial_{pq} \tilde{g}_{ij}\right) g^{k\ell}$$

$$= g^{ij} \tilde{R}_{ijkl} g^{\ell q} \tilde{g}_{ij} g^{\ell q} \tilde{g}_{ij} - g^{ij} \tilde{g}_{\ell ij} g^{k\ell} + g^{ij} g^{k\ell} (\partial_p \tilde{g}_{ij}) g^{pq} (\partial_{k} \tilde{g}_{ij})$$

$$\partial_t (g^{ij} \tilde{g}_{ij}) = -g^{ij} \tilde{g}_{pq} g^{pq} \tilde{g}_{ij} + g^{ij} \tilde{g}_{ij}.$$

Combining the above two identities, we get the identity. \qed

Recall that $\tilde{g} = g_0 - t\psi$. So we get

$$(\partial_t - \Delta_t) \log tr_\omega \tilde{\omega}_t = \frac{(\partial_t - \Delta_t) tr_\omega \tilde{\omega}_t}{tr_\omega \tilde{\omega}_t} + \frac{|\nabla (tr_\omega \tilde{\omega}_t)|^2}{(tr_\omega \tilde{\omega}_t)^2}$$

$$= -\frac{tr_\omega (\psi) + \tilde{R}_{ijkl} g^{ij} g^{k\ell}}{g^{ij} \tilde{g}_{ij}} - \frac{g^{ij} \tilde{g}_{ij} \tilde{g}_{ij} \tilde{g}_{ij} \tilde{g}_{ij} \tilde{g}_{ij}}{g^{ij} \tilde{g}_{ij}} + \frac{g^{ij} \tilde{g}_{ij} \tilde{g}_{ij} \tilde{g}_{ij} \tilde{g}_{ij} \tilde{g}_{ij}}{(g^{ij} \tilde{g}_{ij})^2}$$

We deal with each term on the right-hand-side:
1. \( tr_{\omega_t} \psi \geq -tr_{\omega_t}(C_1 \hat{\omega}_t) = -C_1 tr_{\omega_t} \hat{\omega}_t \).

2. We can simultaneously diagonalize: \( \hat{g}_{ij} = \delta_{ij} \) and \( g_{ij} = \mu_i \delta_{ij} \).

\[
\frac{\hat{R}_{ijkl} g^{ij} g^{kl}}{g^{ij} \hat{g}_{ij}} = \frac{\sum_{i,k} \hat{R}_{ijkl} \mu_{k}^{-1} \mu_{i}^{-1}}{\sum_{i} \mu_{i}^{-1}} \leq K \sum_{k} \mu_{k}^{-1} = K \cdot tr_{\omega_t} \hat{\omega}_t.
\]

3. The last two terms combined is equal to:

\[
-\frac{\mu_{i}^{-1} \mu_{k}^{-1} T_{iik} T_{pik}}{\sum_{i} \mu_{i}^{-1}} + \frac{\mu_{i}^{-1} T_{iiip} \mu_{k}^{-1} T_{kkjl} \mu_{p}^{-1}}{(\sum_{i} \mu_{i}^{-1})^2}
\]

The miracle is that this is non-positive:

\[
\sum_{p,i} \mu_{p}^{-1} |T_{iiip}|^2 = \sum_{p} \mu_{p}^{-1} \sum_{i} |\mu_{i}^{-1/2} T_{iiip} \mu_{i}^{-1/2}|^2 
\leq \sum_{p} \mu_{p}^{-1} \sum_{i} \mu_{i}^{-1} \sum_{j} \mu_{j}^{-1} |T_{jjlp}|^2 
\leq \left( \sum_{p,j,l} \mu_{p}^{-1} \mu_{j}^{-1} |T_{jjlp}|^2 \right) \left( \sum_{i} \mu_{i}^{-1} \right)
\]

Combining the above estimate, we get:

\[(\partial_t - \Delta_t) \log(tr_{\omega_t} \hat{\omega}_t) \leq C_1 + K tr_{\omega_t} \hat{\omega}_t.\]

Since \( \Delta_t u = tr_{\omega_t}(\omega_t - \hat{\omega}_t) = n - tr_{\omega_t} \hat{\omega}_t \), we get:

\[(\partial_t - \Delta_t) (\log(tr_{\omega_t} \hat{\omega}_t) - \lambda \phi) \leq C + (K - \lambda) tr_{\omega_t} \hat{\omega}_t 
= C + (K - \lambda) ((tr_{\omega_t} \hat{\omega}_t) e^{-\lambda \phi}) e^{\lambda \phi}
\]

By choosing \( \lambda = K + 1 \) and \( f(t) = \max_{x \in X} (\log(tr_{\omega_t} \hat{\omega}_t) - \lambda \phi) \), then

\[f'(t) \leq C_1 - C_2 e^{\lambda \phi} \Rightarrow e^{-f} f'(t) - C_1 e^{-f} \leq -C_2
\]

So

\[-\frac{d}{dt} (e^{C_1 t} e^{-f}) \leq -C_2 e^{C_1 t} \Rightarrow e^{C_1 t} e^{-f(t)} \geq C_2 (e^{C_1 t} - 1)/C_1.
\]

So

\[f(t) \leq \log(C_1/C_2) + C_1 t - \log(e^{C_1 t} - 1) = C(t).
\]
So we get:

\[
\log(tr_{\omega_t}\hat{\omega}_t) - \lambda \varphi \leq C(t) \quad \Rightarrow \quad tr_{\omega_t}\hat{\omega}_t \leq C(t)e^{\lambda \varphi} \leq C'(t).
\]

Now because \(\omega^n_t = e^{\partial \varphi / \partial t} \Omega\), we get

\[
C_i \hat{\omega}_t \leq \omega_t \leq C'_i \hat{\omega}_t.
\]

\[\square\]

3 Finite time singularity

3.1 \(C^0\)-estimate

Notice that the above discussion depends on choosing \(\hat{\omega}_t\) for \(t \in [0, T + \epsilon)\). Now that we know the maximal existence time, we can choose any smooth form \(\hat{\omega}_{\max} = \omega_0 - t\psi \in [\omega_0] + T_{\max}2\pi c_1(M)\) with \(\psi = Ric(\Omega)\) without assuming its positivity. We can still define:

\(\hat{\omega}_t = \omega_0 - t\hat{\omega}_{\max}\).

Again in general we don’t have the positivity of \(\hat{\omega}_t\). However we still get the Monge-Ampère flow:

\[
\frac{\partial}{\partial t}\varphi = \log \left(\frac{\hat{\omega}_t + \sqrt{-1}\partial \bar{\partial} \varphi}{\Omega}\right), \quad \varphi(x, 0) = 0.
\]

By exercise 2, one always has upper bound \(\dot{\varphi} \leq C\) for finite time singularity. By integrating we get uniform upper bound \(\varphi \leq CT\).

If there exists a smooth semi-positive form \(\hat{\omega}_{\max} \in [\omega_0] + T_{\max}2\pi c_1(K_M)\), then we can get uniform bound of \(u\). To see this, we can choose \(\epsilon = T_{\max} - T\). The same calculation in (12) shows that

\[
\left(\frac{\partial}{\partial t} - \Delta_t\right)((T_{\max} - t)\dot{\varphi} + \varphi + nt) = tr_{\omega_t}(\hat{\omega}_{\max}) \geq 0.
\]

So by maximal principle, we get:

\[
(T_{\max} - t)\dot{\varphi} + \varphi + nt \geq T_{\max} \min_{x \in M} \dot{\varphi}(x, 0) \geq -C.
\]

So

\[
\varphi \geq -C - nt - (T_{\max} - t)\dot{\varphi} \geq -C'.
\]

More generally, we have:
**Definition 5.** Let $[\alpha]$ be a nef $(1,1)$ class. We say the closed positive current with minimal singularities in the class $[\alpha]$ has bounded potential, if the following condition is satisfied. There exists a constant $C_0 > 0$ such that for every $\epsilon > 0$ there exists $\eta \in C^\infty(X, \mathbb{R})$ such that $\alpha + \sqrt{-1} \partial \bar{\partial} \eta \geq -\epsilon \omega_0$ and $\sup_X |\eta| \leq C_0$.

**Theorem 3.** If the closed positive current with minimal singularities in the class $[\omega_{T_{\max}}]$ has bounded potential, then the KRF solution has uniform lower bound $u(x, t) \geq -C$.

**Proof.** We compute as in (11) and (12) to get:

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) (T_{\max} - t) \dot{\phi} + \varphi + nt) - \epsilon (t \dot{\phi} - \varphi - nt) - \eta = tr_{\omega_t} (\omega_0 - T_{\max} \psi + \epsilon \omega_0 + \sqrt{-1} \partial \bar{\partial} \eta) \geq 0.$$ 

So by minimum principle, we get:

$$(T_{\max} - t - \epsilon t) \dot{\phi} + (1 + \epsilon) \varphi \geq \eta - C \geq -C.$$ 

Because $\dot{\varphi} \leq C$ we get $\varphi \geq -C$ on $X \times [0, T)$.

**Exercise 3.** Conversely if $u \geq -C$ on $X \times [0, T_{\max})$, then the positive closed current with minimal singularities in $[\omega_{T_{\max}}]$ has bounded potential.

**Question:** Under what condition does $[\omega_{T_{\max}}]$ satisfy the condition: positive closed current with bounded potential? This question is open in general.

**Theorem 4.** If $X$ is projective and $[\omega_0] = 2\pi c_1(L)$, then $[\omega_{T_{\max}}]$ contains a smooth semi-positive form and hence satisfy the condition.

This follows from Kawamata’s base-point-free theorem.

**Theorem 5.** Assume $L$ is nef and $L - K_X$ is nef and big. For $m \gg 1$, $mL$ is base point free.

### 3.2 Higher regularity of the limit

**Proposition 2.** The following set is equivalent:

$$\{x \in X| \exists U \ni x \text{ open}, \exists C > 0, \text{ s.t. } |Rm(t)|_{\omega(t)} \leq C \text{ on } U \times [0, T)\}$$

$$= \{x \in X| \exists U \ni x \text{ open}, \exists C > 0, \text{ s.t. } R(t) \leq C \text{ on } U \times [0, T)\}$$

$$= \{x \in X| \exists U \ni x \text{ open}, \exists \omega_U \text{ Kähler metric on } U, \text{ s.t. } \omega_t \rightarrow \omega_U \text{ in } C^\infty(U) \text{ as } t \rightarrow T\}.$$
Proof. Set $3 \subset 1 \subset 2$. So just need to show $2 \subset 3$. On the open set $U$,
\[
\frac{\partial}{\partial t} \dot{\varphi} = -R \geq -C \implies \dot{\varphi} \geq -C \implies u \geq -CT.
\]
Letting $\dot{\omega}_t \equiv \omega_0$ in (13), we get
\[
\left(\frac{\partial}{\partial t} - \Delta_t\right) \log \text{tr} \omega_t \omega_0 \leq C \text{tr} \omega_t \omega_0.
\]
So we get
\[
\left(\frac{\partial}{\partial t} - \Delta_t\right) \left(\log \left(\text{tr} \omega_t \omega_0 + C(t \dot{\varphi} - \varphi - nt)\right)\right) \leq C \text{tr} \omega_t \omega_0 - C \text{tr} \omega_t \omega_0 \leq 0. \tag{14}
\]
By maximal principle,
\[
\text{tr} \omega_t \omega_0 \leq e^{C(t \dot{\varphi} - \varphi - nt)} \leq C \text{ on } U.
\]
The higher order estimate can be derived using parabolic version of Krylov-Evan’s theory. So we get $\|\omega_t\|_{C^k} \leq C$ for any $k \geq 1$. □

Denote by $S$ any of the above sets and call $\Sigma := X \setminus S$ the singularity formation set.

**Corollary 1.** For every finite time singularity of the KRF the singularity formation set $\Sigma$ is nonempty and we have $\limsup_{t \to T} R(t) = +\infty$.

For a nef class $[\alpha] \in H^{1,1}(X, \mathbb{R})$, define
\[
\text{Null}(\alpha) = \bigcup_z \left\{ Z \mid Z \text{ subvariety of } X, \int_Z \alpha^{\dim(Z)} = 0 \right\}.
\]

**Theorem 6 (Collins-Tosatti).** Let $[\alpha] = [\omega_{T_{\max}}]$. Then $\Sigma = \text{Null}(\alpha)$.

Proof. Consider $w = (T_{\max} + \delta - t) \dot{\varphi} + \varphi - nt - \phi$ where $\phi$ is from the next Theorem 7. Then as in (12), we get:
\[
\left(\frac{\partial}{\partial t} - \Delta_t\right) w = \text{tr} \omega_t \left(\omega_0 - (T_{\max} + \delta)\psi + \sqrt{-1} \partial \bar{\partial} \phi\right)
\leq \text{tr} \omega_t \left(\omega_0 - T_{\max} \psi + \sqrt{-1} \partial \bar{\partial} \phi - \delta \psi\right) \geq 0.
\]
So by minimum principle, $\min w$ is increasing. So
\[
(T_{\max} + \delta - t) \dot{\varphi} + \varphi - nt - \phi \geq -C.
\]
So we get lower estimate of \( \dot{\varphi} \):
\[
\dot{\varphi} \geq \frac{1}{T_{\text{max}} + \delta - t} (-u + nt + \phi - C) \geq \frac{1}{\delta} \dot{\varphi} - C \delta. \tag{15}
\]
The partial \( C^2 \)-estimate can be done as in (14). Higher order estimate follows from Krylov-Evans.

The above proof depends on the following construction of barrier functions.

**Theorem 7.** Let \([\alpha]\) be a closed real \((1,1)\) class such that \(\alpha\) is nef and \(\int_X \alpha^n > 0\). Then there exists an upper semi-continuous \(L^1\) function \(\phi : X \to \mathbb{R} \cup \{-\infty\}\), which equals \(-\infty\) on \(\text{Null}(\alpha)\), which is finite and smooth on \(X \setminus \text{Null}(\alpha)\) and such that
\[
\alpha + \sqrt{-1} \partial \bar{\partial} \phi \geq \epsilon \omega_0.
\]
on \(X \setminus \text{Null}(\alpha)\) for some \(\epsilon > 0\).

Theorem 7 generalizes following theorem is known due to Kodaira’s lemma.

**Theorem 8.** Let \(D\) be a nef and big divisor on a projective manifold \(X\). Then there is an effective \(\mathbb{R}\)-divisor \(E\) such that \(D - E\) is ample.

We can choose \(e^{-\phi} = \frac{e^{-\varphi}}{|\sigma|^2} = \frac{e^{-\varphi} e^{-\phi_1}}{\|\sigma\|_h^2}\) where \(e^{-\varphi}\) is a positively curved metric on \(D - E\) and \(\frac{1}{|\sigma|^2}\) is a singular metric on \(E\) where \(E = \{\sigma = 0\}\).

\[
\sqrt{-1} \partial \bar{\partial} \phi = \sqrt{-1} \partial \bar{\partial} (\varphi + \phi_1) + \log \|\sigma\|_h^2 \geq \sqrt{-1} \partial \bar{\partial} \varphi \geq \epsilon \omega_0.
\]

If we define augmented base locus (no-ample locus):
\[
\mathbf{B}_+(D) = \bigcap_E \{E; E\text{ effective and } D - E\text{ ample}\}
\]
\[
= \bigcap_{m \in \mathbb{N}} \mathbf{B}(mD - A) = \bigcap_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} B_s[k(mD - E)].
\]

Then actually \(\mathbf{B}_+(D) = \text{Null}(D)\) (Nakamaye’s theorem).

**Theorem 9.** If \( [\omega_0] = 2\pi c_1(L) > 0 \). Then \(\omega_t\) converges smoothly to a Kähler metric \(\omega_{T_{\text{max}}}\) on \(X \setminus \mathbf{B}_+(D)\) where \(D = L + T_{\text{max}}K_X\).

**Remark 3.** There is an analytic characterization of the augmented base locus. One can define the non-Kähler locus of a \((1,1)\)-class \([\alpha]\) as:
\[
E_{nK}(\alpha) = \bigcap_{T \in [\alpha]} \text{Sing}(T)
\]
where the intersection ranges over all Kähler currents \(T = \alpha + \sqrt{-1} \partial \bar{\partial} \varphi\) in the class \([\alpha]\). Then Boucksom showed that \(\mathbf{B}_+(D) = E_{nK}(D)\), and moreover there is a Kähler current with analytic singularities along \(E_{nK}(D)\).
4 Normalized Kähler-Ricci flow on Fano manifolds

A special finite time singularity happens when $X$ is a Fano manifold and $[ω_0] = 2πc_1(X) > 0$. In this case, the maximal existence time $T_{\text{max}} = 1$ and the $X$ converges in the Gromov-Hausdorff sense to a point as $t → 1$. To get more information, we consider the normalized Kähler-Ricci flow. Assume $ω_t$ is a solution to the unnormalized KR flow. Define $\tilde{ω} = \omega / (1 - t)$ so that $\text{Vol}(\tilde{ω})$ is fixed. Then

$$\frac{\partial}{\partial t} \tilde{ω} = \frac{ω}{(1 - t)^2} + \frac{1}{1 - t} \frac{∂ω}{∂t} = \frac{1}{1 - t} (\tilde{ω} - \text{Ric}(ω)).$$

Notice that $\text{Ric}(ω_t) = \text{Ric}(\tilde{ω}_t)$. If we define $\tilde{t} = -\log(1 - t)$, then we get the normalized Kähler-Ricci flow

$$\frac{∂}{∂\tilde{t}} \tilde{ω} = -\text{Ric}(\tilde{ω}) + \tilde{ω}, \quad \tilde{ω}(0) = ω_0. \quad (16)$$

This flow exists for any time $t ∈ [0, ∞)$. Let $\tilde{ω}_t$ be the solution to the normalized KRF (16) and $\tilde{g}_t$ be the associated Kähler metric. If $|Rm(\tilde{g}(t))| < ∞$ and the diameter of $\tilde{g}(t)$ is uniformly bounded, then $\tilde{g}_t$ converges in the Cheeger-Gromov sense to a metric $\tilde{g}_∞$ on $X$.

**Proposition 3.** If $|Rm(\tilde{g}(t))|$ and $\text{diam}(\tilde{g}(t))$ is uniformly bounded, then the Cheeger-Gromov limit $\tilde{g}_∞$ is a Kähler-Ricci soliton.

**Proof.** 1. Recall Perelman’s W-entropy and $μ$-energy:

$$W(g, f, τ) = \int_X \left( τ(R + |∇f|^2) + f - n \right) (4πτ)^{-n/2} e^{-f} dV_g.$$  

Under the evolution equation:

$$∂_s g = -2\text{Ric}(g), \quad ∂_s f = -\Delta f + |∇f|^2 - R + \frac{n}{2τ}, \quad ∂_s τ = -1.$$  

We have

$$\frac{d}{ds} W(g(s), f(s), τ(s)) = \int_X 2τ \left| R_{ij} + f_{ij} - \frac{1}{2τ} g_{ij} \right|^2 (4πτ)^{-n/2} e^{-f} dV_g.$$  

The $μ$-energy:

$$μ(g, τ) = \inf \left\{ W(g, f, τ); \int_X (4πτ)^{-n/2} e^{-f} dV_g = 1 \right\}. \quad (17)$$
Then $\mu(g(s), \tau(s))$ is increasing. It’s strictly increasing unless $g(s)$ is a Kähler-Ricci soliton.

Notice that we have the rescaling invariance: $W(g, f, \tau) = W(\tau^{-1}g, f, 1)$ and $\mu(g, \tau) = \mu(\tau^{-1}g, 1)$. To adapt to the our setting, we let $t = 2s$, $\tau(s) = \frac{1}{2} - s = \frac{1}{2}(1 - t)$, $\bar{g}(t) = (1 - t)^{-1}g = (2\tau)^{-1}g$. Then

$$\frac{d}{dt} W\left(\bar{g}(t), f, \frac{1}{2}\right) = \frac{1}{2} \frac{d}{ds} W(g(s), f(s), \tau(s))$$

$$= \frac{1 - 2s}{2} \int_X \left| \bar{R}_{ij} + f_{ij} - g_{ij} \right|^2 (2\pi)^{-n/2} e^{-f} dV_{\bar{g}}$$

So if $\check{t} = -\log(1 - t) = -\log(1 - 2s)$, then we get:

$$\frac{d}{dt} W\left(\bar{g}(\check{t}), f(\check{t}), \frac{1}{2}\right) = \int_X \left| \bar{R}_{ij} + f_{ij} - \bar{g}_{ij} \right|^2 (2\pi)^{-n/2} e^{-f} dV_{\bar{g}}$$

under the evolution equation:

$$\frac{\partial \bar{g}}{\partial t} = -\text{Ric}(\bar{g}) + \bar{g}, \quad \frac{\partial f}{\partial t} = -\bar{\Delta}f + |\nabla f|_{\bar{g}}^2 - \bar{R} + n.$$  

2. For simplicity, we will deal with the normalized functional and remove the tilde symbol. Let $f_{i+1}$ be the minimizer of $W(g(t_i + A), f, 1/2)$. Then

$$0 \leq W(g(t_i + A), f_{i+1}, 1/2) - W(g(t_i), f, 1/2)$$

$$\leq \int_0^A \int_X \left| R_{jk} + \nabla_j \nabla_k f_{i+1}(t_i + s) - g_{jk} \right|^2 dV_{t_i+s} ds +$$

$$\int_0^A \int_X (|\nabla_j \nabla f_{i+1}(t_i + s)|^2 + |\nabla_j \nabla_k f_{i+1}(t_i + s)|^2) dV_{t_i+s} ds$$

$$\leq \mu(g(t_i + A), 1/2) - \mu(g(t_i), 1/2).$$

$\mu(g_{i+1}, 1/2) \rightarrow \mu(g_{i+1}, 1/2)$ is uniformly bounded. So one can show that as $i \rightarrow +\infty$, $f_{i+1}(s)$ converges to $f_{\infty}(s)$ that is a weak solution to the equation:

$$R_{jk}(g_{\infty}(s)) + (f_{\infty}(s))_{jk} - (g_{\infty})_{jk} = 0, \quad (f_{\infty}(s))_{jk} = (f_{\infty}(s))_{jk} = 0.$$  

By elliptic regularity, $f_{\infty} \in C^\infty(X)$ so $g_{\infty}$ is a Kähler-Ricci soliton.  

$\square$
The assumption on the Riemannian curvature and diameter are used to make sure there is a convergence sequence in the Cheeger-Gromov sense. It turns out that the diameter is always bounded by Perelman’s work.

**Theorem 10 (Perelman).** Let \( g(t) \) be a normalized Kähler-Ricci flow on a Fano manifold \( X \). There exists a uniform constant \( C \) so that

- \( |R(g(t))| \leq C \).
- \( \text{diam}(X, g(t)) < +\infty \).

1. (Get a uniform lower bound of the Ricci potential \( u \)). Let \( \omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi \) and \( \text{Ric}(\omega(t)) - \omega(t) = -\sqrt{-1} \partial \bar{\partial} \varphi \). Then the normalized KRF is equivalent to:

\[
\frac{\partial}{\partial t} \phi(t) = u(t), \quad \phi(0) = 0.
\]  
(18)

Notice that we can write:

\[
\text{Ric}(\omega(t)) - \omega(t) = \text{Ric}(\omega(t)) - \text{Ric}(\omega_0) + \text{Ric}(\omega_0) - \omega_0 + \omega_0 - \omega(t)
\]

\[
= -\sqrt{-1} \partial \bar{\partial} \log \frac{\omega(t)^n}{\omega_0^n} - \sqrt{-1} \partial \bar{\partial} \varphi_0 - \sqrt{-1} \partial \bar{\partial} \phi.
\]

So the NKRF is equivalent to

\[
\frac{\partial}{\partial t} \phi(t) = \log \frac{\omega(t)^n}{\omega_0^n} + \phi(t) + u(0) + c_t, \quad \phi(0) = 0.
\]  
(19)

Since \( u \) is defined up to a constant, we can normalize \( u(t) \) so that:

\[
\int_X e^{-u} \omega(t)^n = (2\pi)^n.
\]  
(20)

Notice that \( n - R = \Delta u \) with \( n = 2m \).

\[
W\left( g(t), u, \frac{1}{2} \right) = \int_X \left[ (R + |\nabla u|^2) + u - m \right] (2\pi)^{-m/2} e^{-u} \omega(t)^n
\]

\[
= \int_X \left[ (n - \Delta u + |\nabla u|^2) + u - 2n \right] (2\pi)^{-n} e^{-u} \omega^n
\]

\[
= (2\pi)^{-n} \int_X u e^{-u} \omega^n - n.
\]

Because \( \mu(g, 1/2) = \inf\{W(g, f, 1/2); \int_X (2\pi)^{-n} e^{-f} \omega(t)^n\} \) is increasing, we get \( W(g(t), u, 1/2) \geq -C \). In other words, we have:
Lemma 3. There is a uniform constant $C_1 = C_1(\mu(g(0), 1/2)) > 0$ so that $\int_X u e^{-u} \omega^n \geq -C_1$.

Proof. Write $u = u_+ + u_-$ with $u_+ = \max\{0, u\}$ and $u_- = \min\{0, u\}$. The lemma follows from the fact that $x e^{-x}$ is a bounded function for $x \geq 0$. \qed

Lemma 4. The scalar curvature $R(t)$ is uniformly bounded from below along the flow.

Proof. Recall that along the un-normalized flow, we have:
\[
\frac{\partial}{\partial s} \hat{R}(s) = \hat{\Delta}_{\text{real}} \hat{R}(s) + 2|\text{Ric}(\hat{g}(s))|^2. \tag{21}
\]
Then by minimum principle, $\hat{R}(s)$ is uniformly bounded from below. For normalized flow, $g(s) = (1 - 2s)^{-1}\hat{g}(s)$. $R(g(s)) = (1 - 2s)R(\hat{g}(s))$ is also uniformly bounded from below. $R(g(s))$ satisfies:
\[
\frac{\partial}{\partial s} R(s) = -2 R(\hat{g}(s)) + (1 - 2s) \left( \hat{\Delta} \hat{R}(s) + 2|\text{Ric}(\hat{g}(s))|^2 \right).
\]
This implies for $t = -\log(1 - 2s)$, we have:
\[
\frac{\partial}{\partial t} R(t) = \frac{1}{2} \Delta_{\text{real}} R(t) - R(t) + |\text{Ric}(g(t))|^2_{\text{real}} \tag{22}
\]
Notice that $\Delta_{\text{real}} f = 2\Delta f = g^{ij} (f_{ij} + f_{ji})$. \qed

The Ricci potential $u(t) = \frac{\partial}{\partial t} \phi(t)$ satisfies:
\[
\sqrt{-1} \partial \bar{\partial} \phi = \frac{\partial}{\partial t} \sqrt{-1} \partial \bar{\partial} \phi = \sqrt{-1} \partial \bar{\partial} \frac{\partial}{\partial t} \phi
\]
\[
= \sqrt{-1} \partial \bar{\partial} tr_{\omega(t)} (-\text{Ric}(\omega(t)) + \omega(t)) + \sqrt{-1} \partial \bar{\partial} \phi
\]
\[
= \sqrt{-1} \partial \bar{\partial} (\Delta u + u).
\]
Define $a = -(2\pi)^{-n} \int_X u e^{-u} dV$. This implies
\[
\frac{\partial}{\partial t} u = \Delta u + u + a \tag{23}
\]
where $a = -\int_X u e^{-u} (2\pi)^n \leq C$ by the previous lemma.
**Lemma 5.** The function $u(t)$ is uniformly bounded from below.

**Proof.** Prove by contradiction. If the Ricci potential $u(t)$ is very negative for some time $t_0$ say $u(t) \leq -2(n + C_1)$. By $C_1$ very big, then

$$\frac{\partial}{\partial t} u = n - R + u + a \leq -(n + C_1) - R < 0. \quad (24)$$

This implies $u(t)$ stays very negative for $t \geq t_0$. If at $t_0$, $u(y_0) \ll 0$, then $u(y) \ll 0$ for all $y \in U$ a neighborhood of $y_0$. So all $t \geq t_0$, $u(y) \ll 0$ for $y \in U$. Then $\frac{\partial}{\partial t} u \leq C + u$ implies

$$u(t)(y) \leq e^{t-t_0}(C + u(t_0)) \leq -\tilde{C}e^t \text{ for } t \geq t_0. \quad (25)$$

Then $\dot{\phi} = u$ yields

$$\phi(t)(y) \leq \phi(t_0)(y) - \tilde{C}e^{t-t_0} \leq -C_1e^t. \quad (26)$$

Because $\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$, we have

$$\Delta_0 \phi(t) = -tr_\omega(t)\omega(t) + n \leq n.$$

Assume $\phi(x,t) = \max_M \phi(y,t)$ and $G_0$ is the Green’s function associated to the metric $g(0)$. By Green’s formula, we have:

$$\phi(x,t) = \frac{1}{Vol_0(M)} \int_M \phi(y,t) dV_0 - \frac{1}{Vol_0(M)} \int_M \Delta_0 \phi(y,t) G_0(x,t, y) dV_0$$

$$\leq \frac{Vol_0(M \setminus U)}{Vol_0(M)} \sup_M \phi(\cdot, t) + \frac{Vol_0(U)}{Vol_0(M)} \int_U \phi(y,t) dV_0 + C$$

$$\leq \frac{Vol_0(M \setminus U)}{Vol_0(M)} \sup_M \phi(\cdot, t) - Ce^t + C.$$  

for $t \geq t_0$. Since $Vol_0(M \setminus U)/Vol_0(M) < 1$, we get:

$$\max_M \phi(\cdot, t) \leq -Ce^t + C. \quad (27)$$

On the other hand we can estimate $\phi(\cdot, t)$ from below. Let $u(z_t, t) = \max_M u(t)$. Because $\int_M e^{-u} dV_t = (2\pi)^n / n$, $u(z_t, t) \geq -C$. We also have:

$$\frac{d}{dt}(u(t) - \phi(t)) = \Delta u + a = n - R + a. \quad (28)$$
Because $R$ is uniformly bounded from below by Lemma 4, we get:

$$u(z_t, t) - \phi(z_t, t) \leq \max_M (u(\cdot, 0) - \phi(\cdot, 0)) + Ct.$$  

This implies

$$\max_M \phi(t) \geq \phi(z_t, t) \geq u(z_t, t) - Ct \geq -C - Ct.$$  

This contradicts (27). \qed

2. Bound $|\nabla u|$ and $R(t) = n - \Delta u$ by $C^0$-norm of Ricci potential $u(t)$. This can be achieved by considering evolution equation for $\frac{|\nabla u|^2}{u + 2B}$ and $\frac{-\Delta u}{u + 2B}$, where $B$ is a uniform constant such that $u + B > 0$.

**Proposition 4.** There is a uniform constant $C$ so that:

$$|\nabla u|^2 \leq C(u + C), \quad R \leq C(u + C). \quad (29)$$

**Proof.** We first calculate:

$$\square |\nabla u|^2 = \left( \frac{\partial}{\partial t} - \Delta \right) |\nabla u|^2 = -u_{ij} u_{ij} - u_{ij} u_{ji} + u_i u_i$$

$$= -|\nabla \nabla u|^2 - |\nabla \nabla u|^2 + |\nabla u|^2.$$  

Let $H = \frac{|\nabla u|^2}{u + 2B}$, then we can calculate:

$$\square H = \frac{-|\nabla \nabla u|^2 - |\nabla \nabla u|^2}{u + 2B} + \frac{|\nabla u|^2(2B - a)}{(u + 2B)^2} \quad \frac{\nabla |\nabla u|^2 \cdot \nabla u + \nabla |\nabla u|^2 \cdot \nabla u}{(u + 2B)^2} - \frac{2|\nabla u|^4}{(u + 2B)^3}.$$  

Write:

$$\frac{\nabla |\nabla u|^2 \cdot \nabla u + \nabla |\nabla u|^2 \cdot \nabla u}{(u + 2B)^2} - \frac{2|\nabla u|^4}{(u + 2B)^3} = (1 - \epsilon) \frac{\nabla u \cdot \nabla H + \nabla u \cdot \nabla u}{u + 2B} + \epsilon \left[ \frac{\nabla |\nabla u|^2 \cdot \nabla u + \nabla |\nabla u|^2 \cdot \nabla u}{(u + 2B)^2} - \frac{2|\nabla u|^4}{(u + 2B)^3} \right].$$  

We estimate:

$$|u_i(u_j u_j)| = |u_i u_{ji} u_j + u_i u_j u_{ji}| \leq |\nabla u|^2(|\nabla \nabla u|^2 + |\nabla \nabla u|^2).$$
So
\[
\frac{\nabla u \cdot \nabla |\nabla u|^2}{(u + 2B)^2} \leq \frac{\epsilon |\nabla u|^2(\nabla \nabla u + |\nabla \nabla u|)}{(u + 2B)^{3/2}(u + 2B)^{1/2}} = \frac{\epsilon |\nabla u|^4}{2(u + 2B)^3} + 4\epsilon |\nabla \nabla u|^2 + |\nabla \nabla u|^2 \frac{u + 2B}{u + 2B}.
\]

Choose \(\epsilon\) sufficiently small, we get:
\[
\square H \leq \frac{|\nabla u|^2(2B - a)}{(u + 2B)^2} + (1 - \epsilon) \frac{\nabla u \cdot \nabla H + \nabla u \cdot \nabla H}{u + 2B} - \frac{\epsilon |\nabla u|^4}{2(u + 2B)^3}.
\]

By maximal principle, we get:
\[
\frac{d}{dt} H_{\text{max}} \leq \frac{|\nabla u|^2}{(u + 2B)^2} \left(2B - a - \frac{\epsilon |\nabla u|^2}{2u + 2B}\right). \tag{30}
\]

\textbf{Corollary 2.} This is a uniform constant \(C\) such that
\[
u(y,t) \leq C \text{dist}_t^2(x,y) + C, \quad R(y,t) \leq C \text{dist}_t^2(x,y) + C, \quad |\nabla u| \leq C \text{dist}_t^2(x,y) + C. \tag{31}
\]

\textbf{Proof.} By Lemma 5, we can assume \(u \geq \delta > 0\). From (29), it follows that \(\sqrt{u}\) is Lipschitz function. Therefore
\[
|\sqrt{u}(y,t) - \sqrt{u}(z,t)| \leq C \text{dist}_t(y,z).
\]

Choose \(z\) such that \(u(z,t) = \min_M u(\cdot, t)\). Then \(u(z,t) \leq K\) for a constant that does not depend on \(t\). So we get \(u(y,t) \leq C \text{dist}_t^2(y,z) + C\). The other two estimates follow from (29).

\square

3. (bound the diameter)

\textbf{Lemma 6.} For every \(\epsilon > 0\), we can find \(B(2^{k_1}, 2^{k_2})\), such that if \(\text{diam}(M, g(t))\) is large enough, then
\[
\text{Vol}(B(2^{k_1}, 2^{k_2})) < \epsilon \text{ and } \text{Vol}(B(2^{k_1}, 2^{k_2})) \leq 2^{10n}\text{Vol}(B(k_1 + 2, k_2 - 2)). \tag{32}
\]
Proof. The first estimate is easy. To get the second estimate, we first need to show that:

\[ \text{Vol}(B(2^k, 2^{k+1})) \geq 2^{2k}2^{-kn}C. \]  \hspace{1cm} (33)

To see this, first we have \( R \leq C2^{2k} \) on \( B(2^k, 2^{k+1}) \) due to (31). Then (33) follows from the local noncollapsing property along the Ricci flow: \( \text{Vol}(B(x, 2^{-k})) \geq 2^{2k}C \) for any \( B(x, 2^{-k}) \subset B(2^k, 2^{k+2}) \setminus B(2^{k+2}, 2^{k-2}). \)

**Lemma 7.** There exist \( r_1, r_2 \) and a uniform constant \( C \) such that \( 2^{k_1} \leq r_1 \leq 2^{k_1+1}, 2^{k_2} \leq r_2 \leq 2^{k_2+1} \) such that

\[ \int_{B(r_1, r_2)} R \leq C \cdot \text{Vol}(B(2^{k_1}, 2^{k_2})). \]

**Proof.** By mean value theorem, we know that there exists \( r_1 \in [2^{k_1}, 2^{k_1+1}] \) and \( r_2 \in [2^{k_2-1}, 2^{k_2}] \) such that

\[
\text{Vol}(S(r_1)) \leq \frac{\text{Vol}(B(2^{k_1}, 2^{k_2}))}{2^{k_1+1}}, \quad \text{Vol}(S(r_2)) \leq \frac{\text{Vol}(B(2^{k_1}, 2^{k_2}))}{2^{k_2+1}}.
\]

So we can estimate (\( V = \text{Vol}(B(2^{k_1}, 2^{k_2})) \)):

\[
\int_{B(r_1, r_2)} R = \int_{B(r_1, r_2)} (R - n) + n\text{Vol}(B(r_1, r_2))
\]
\[
= -\int_{B(r_1, r_2)} \Delta u + n\text{Vol}(B(r_1, r_2))
\]
\[
\leq \int_{S(r_1)} |\nabla u| + \int_{S(r_2)} |\nabla u| + n\text{Vol}(B(r_1, r_2))
\]
\[
\leq \frac{V}{2^{k_1}2^{k_1+1}} + \frac{V}{2^{k_2}} C 2^{k_2+1}
\]
\[
= CV
\]

**Proposition 5.** There is a uniform constant \( C \) such that \( \text{diam}(M, g(t)) \leq C. \)

**Proof.** Proof by contradiction. Assume \( \text{diam}(M, g(t)) \) is not uniformly bounded in \( t \). Then there exists a sequence \( t_i \to +\infty \) such that \( \text{diam}(M, g(t_i)) \to +\infty \). Let \( \epsilon_i \to 0 \). By Lemma 6, there exist sequences \( k_1^i \) and \( k_2^i \) such that

\[
\text{Vol}_{t_i}(B_{t_i}(2^{k_1^i}, 2^{k_2^i})) < \epsilon_i, \quad \text{Vol}(B_{t_i}(2^{k_1^i}, 2^{k_2^i})) \leq 2^{10n}\text{Vol}(B_{t_i}(2^{k_1^i+2}, 2^{k_1^i-2})). \]  \hspace{1cm} (34)
By Lemma 7, there exists $r_1^i \in [2^{k_i}, 2^{k_i+1}]$ and $r_2^i \in [2^{k_i-1}, 2^{k_i}]$ such that

$$\int_{B_{t_i}(r_1^i, r_2^i)} R \leq CVol(B(k_1^i, k_2^i)).$$

Let $\eta = \eta_i$ be a sequence of cut-off functions such that $\eta(z) = 1$ for $z \in [2^{k_i+2}, 2^{k_i-2}]$ and equal to 0 for $z \in (-\infty, r_1^i] \cup [r_2^i, \infty)$. Let $w_i(x) = e^{C_i \eta_i(dist_t_i(x, p_i))}$ such that $(2\pi)^{-n} \int_M w_i^2 = 1$. Let $w^2 = (2\pi)^{-n} e^{-f}$ so that $f = -2 \log w - n \log 2\pi$. We estimate each term of

$$W(g, f, 1/2) = \int_M (R + |\nabla f|^2 + f - 2n)(2\pi)^{-n} e^{-f} dV_g$$

$$= \int_M (Ru^2 + 4|\nabla w|^2 - w^2 \log w^2) dV_g - C.$$

(a)

$$\int_M R_t w_i^2 \leq e^{2C_i} \int_{B(r_1, r_2)} r \leq e^{2C_i} Vol(B(2^{k_1}, 2^{k_2}))$$

$$\leq 2^{10n} Vol(B(2^{k_1+2}, 2^{k_2-2})) \lesssim \int_M u^2 dV = 1.$$

(b)

$$\int_M |\nabla w|^2 dV_g = \int_M e^{2C} |\eta'|^2 dV_g = e^{2C} Vol(B(2^{k_1}, 2^{k_2}))$$

$$\leq 2^{10n} e^{2C} Vol(B(2^{k_1+2}, 2^{k_2-2}))$$

$$\leq \int_M u^2 dV = C.$$

$\Box$

5 Long time behavior KRF

Example 2. Consider the product $M = \Sigma_1 \times \Sigma_2$ where $\Sigma_1$ is a torus, and $\Sigma_2$ is a Riemann surface of genus $\geq 2$. Assume $[\omega_0] = [\eta_1 + \eta_2 = (a, b)$ where $a = \langle [\eta_1], [\Sigma_1] \rangle > 0$ and $b = \langle [\eta_2], [\Sigma_2] \rangle > 0$. Because $2\pi c_1(M) = 2\pi c_1(\Sigma_2) = 2\pi(2 - 2g)$. So $[\omega_t] = [\eta_1] + [\eta_2] - t2\pi c_1(\Sigma_2) = (a, b - 2\pi t(2 - 2g))$. So $T_{\text{max}} = \infty$ and the volume $Vol(M, \omega_t) = a + b + 2\pi t(2g - 2)$ blows up. If we consider the normalized metric $\tilde{\omega}_t = \omega_t/t = (a/t, b/t - 2\pi(2 - 2g)) \rightarrow (0, 2\pi(2g - 2))$. One can show that $(M, g_t) \rightarrow (\Sigma_2, g_\infty)$ in the Gromov-Hausdorff sense, where $g_\infty$ is the hyperbolic metric on $\Sigma_2$. 

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Let $\omega_s, s \in [0, \infty)$ be a solution to the Kähler-Ricci flow. Then we know that $K_M$ is nef, i.e. $c_1(K_M) \in \overline{C_M}$. Denote $\omega_s = \hat{\omega}_s / (1 + s)$. Then $\omega_t$ satisfies the equation:

$$\frac{\partial \omega}{\partial s} = \frac{\partial_s \hat{\omega}_s}{1 + s} - \frac{\hat{\omega}_s}{(1 + s)^2} = \frac{1}{1 + s}(-Ric(\omega_s) - \omega_s).$$

Let $t = \log(1 + s)$, we get:

$$\frac{\partial}{\partial t} \omega(t) = -Ric(\omega(t)) - \omega(t), \quad \omega(0) = \omega_0.$$ (35)

The Kähler class evolves like:

$$[\omega(t)] = \frac{[\omega_0] - 2\pi s c_1(M)}{1 + s} = e^{-t}[\omega_0] - (1 - e^{-t})2\pi c_1(M).$$

So the Kähler class converges to the first Chern class of the canonical line bundle:

$$[\omega(t)] \longrightarrow -2\pi c_1(M) = 2\pi c_1(K_M) \text{ as } t \rightarrow +\infty.$$

1. $c_1(M) = 0$. Then $\hat{\omega}_s$ converges to a Ricci flat metric on $M$.

2. $c_1(M)$ is big: $\int_M c_1(M)^n > 0$. Then $\omega(t)$ converges to a singular Kähler-Einstein metric on $M$ which is smooth on $M \setminus \text{Null}(K_M)$.

3. $c_1(M)$ is not big. $\int_M c_1(M)^n = 0$. Then conjecturally $(M, \omega(t))$ converges to a lower dimensional space $N$ of dimension given by:

$$\dim_{\mathbb{C}} N = \max\{k; c_1(M)^k \neq 0\} =: \nu(M).$$

This is essentially the Abundance Conjecture:

**Conjecture 1.** (a) $\kappa(M) = \nu(M)$ where

$$\kappa(M) = \max \left\{ k; \lim sup_{m \rightarrow +\infty} \frac{\dim_{\mathbb{C}} H^0(M, mK_M)}{m^k} > 0 \right\}.$$

(b) $K_M$ is base-point-free. In other words, for $l \gg 1$, $|lK_M|$ gives rise to a morphism $M \rightarrow N$ with $\dim_{\mathbb{C}} N = \kappa(M)$.

**Example 3.** The above example is a trivial elliptic fibration. In general if we have an elliptic fibration $\pi : M \rightarrow \Sigma$. Assume on $\Sigma^0 = \Sigma \setminus \{p_i\}$, the fibration is smooth...
and $m_i$ is the multiplicity of the (possibly singular) fibre $\pi^{-1}(p_i)$. Then we have the canonical bundle formula:

$$K_M = \pi^* \left( K_{\Sigma} + f_1(\mathcal{O}_M)^\vee + \sum_{i=1}^k \left( 1 - \frac{1}{m_i} \right) p_i \right)$$

(36)

Assume $K_M$ is nef. Then $\kappa(M) = 1$ if and only if

$$\delta(\pi) = \chi(\mathcal{O}_X) + \left( 2g - 2 + \sum_{i=1}^k (1 - m_i^{-1}) \right) > 0.$$ 

In this case, one can show that there exists a Kähler metric $\omega_\infty$ on $\Sigma$ satisfying the equation:

$$\text{Ric}(\omega_\infty) = -\omega_\infty + j^* \omega_{WP} + 2\pi \sum_{i=1}^k (1 - m_i^{-1}) \delta_{p_i},$$

where $j : \Sigma^\circ \to M$ is the induced map from the $\Sigma$ to the moduli space of Riemann surfaces of genus 1.

**Theorem 11.** As $t \to +\infty$, $\omega(t)$ converges to $\pi^* \omega_\infty$ as currents. Furthermore, on $\pi^{-1}(\Sigma^\circ)$, $\omega(t)$ converges to $\pi^* \omega_\infty$ uniformly.

In higher dimensions, assuming the Abundance Conjecture, if $0 < \kappa(M) < n$, then there is a fibration $M \to N$ such that the generic fibre is a Calabi-Yau manifold. We have a similar canonical bundle formula. One can define a generalized Kähler-Einstein metric on the base $N$:

$$\text{Ric}(\omega_\infty) = -\omega_\infty + \omega_{WP} \text{ on } N^\circ.$$ 

(37)

To define $\omega_{WP}$, we use the work by Tian-Todorov on smoothness of the deformation space of Calabi-Yau manifolds.

$$\omega_{WP} = \sqrt{-1} \partial \bar{\partial} \log \int_{M_t} \Psi \wedge \bar{\Psi}.$$ 

where $t$ are coordinates on $N^\circ$ and $\Psi$ is a holomorphic section of $H^0(U, K_{M/N})$. Choose a metric $\chi \in 2\pi c_1(M)$. Then (37) can be reduced a Monge-Ampère equation:

$$(\chi + \sqrt{-1} \partial \bar{\partial} \phi)^\kappa = F e^\phi \chi^\kappa.$$ 

(38)
where $F$ is defined in (40). On the other hand, the NKRF in (35) can be reduced to the equation:

$$\frac{\partial}{\partial t} \varphi = \log \frac{e^{(n-\kappa)t}(\hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} - \varphi, \quad \varphi(0) = 0.$$  \hspace{1cm} (39)

where $\hat{\omega}(t) = e^{-t}\omega_0 - (1 - e^{-t})\chi$ and $\Omega$ is a volume form on $M$ satisfying $\text{Ric}(\Omega) = \chi$. So the claim is that the solution to (39) converges to the solution to (38) as $t \to +\infty$. We will only give a heuristic reason why this is true. First we can choose $\omega_0$ such that $\omega_0|_{M_t}$ is Ricci flat and

$$\pi^* F = \frac{\Omega}{\omega_0^{n-\kappa} \wedge \chi^\kappa}. \quad (40)$$

Then

$$e^{(n-\kappa)t}(\hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{(n-\kappa)t}(\hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi)^n \frac{\omega_0^{n-\kappa} \wedge \chi^\kappa}{\Omega} = e^{(n-\kappa)t}(\hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi)^n \frac{\omega_0^{n-\kappa} \wedge \chi^\kappa}{\omega_0^{n-\kappa} \wedge \chi^\kappa} (\pi^* F)^{-1}.$$

So the proof is reduced to show that:

$$e^{(n-\kappa)t}(e^{-t}\omega_0 - (1 - e^{-t})\chi + \sqrt{-1} \partial \bar{\partial} \varphi)^n \omega_0^{n-\kappa} \wedge \chi^\kappa \rightarrow (\chi + \sqrt{-1} \partial \bar{\partial} \varphi)^\kappa \frac{\omega_0^{n-\kappa} \wedge \chi^\kappa}{\chi^\kappa}. $$

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