

# Special test configuration and K-stability of Fano varieties

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## Abstract

For any flat projective family  $(\mathcal{X}, \mathcal{L}) \rightarrow C$  such that the generic fibre  $\mathcal{X}_\eta$  is a klt  $\mathbb{Q}$ -Fano variety and  $\mathcal{L}|_{\mathcal{X}_\eta} \sim_{\mathbb{Q}} -K_{\mathcal{X}_\eta}$ , we use the techniques from the minimal model program (MMP) to modify the total family. The end product is a family such that *every* fiber is a klt  $\mathbb{Q}$ -Fano variety. Moreover, we can prove that the Donaldson-Futaki invariants of the appearing models *decrease*. When the family is a test configuration of a fixed Fano variety  $(X, -K_X)$ , this implies Tian's conjecture: given  $X$  a Fano manifold, to test its K-(semi, poly)stability, we only need to test on the special test configurations.

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This article is motivated by studying Tian's conjecture, which says that to test K-(semi, poly)stability we only need to consider the test configurations whose special fibers are  $\mathbb{Q}$ -Fano varieties. It consists of two parts. In the first part, inspired by Tian's conjecture, we obtain our main result on the existence of special degenerations of Fano varieties. In the second part, we apply the result from the first part to study K-stability of Fano varieties. In particular, we give an affirmative answer to Tian's conjecture.

## Part 1. Family of Fano Varieties

Through out this part, we work over an algebraically closed field that is of characteristic 0.

### 1. Introduction: main results

1.1. *Degenerations of Fano varieties.* For various questions, especially for compactifying the moduli spaces, people are interested in the degenerations of varieties. When the varieties have positive canonical classes, i.e., they are *canonically polarized*, this question has attracted much interests. The one dimensional case, namely, the degeneration of smooth curves of genus at least 2, has been understood well after Deligne-Mumford's work [DM69]. The study of higher dimensional case by an analogous strategy was initiated more than two decades ago (see [KSB88], [Ale96]). Using the recent monumental progress on the minimal model program of [BCHM10] and many other work, the fundamental aspects of this theory are close to being completely settled. See Kollár's survey paper [Kol13] for more details. One essential point of such varieties having a nice moduli theory is that they carry natural polarizations, namely, their canonical classes.

Another class of varieties carrying natural polarization is the class of Fano varieties, whose canonical classes are negative. However, such varieties behave quite differently with the canonically polarized varieties. For example, there exists a family of Fano manifolds, whose general fibers are isomorphic to a given Fano manifold, but the complex structure jumps at the special fibers (see, e.g., [Tia97, §7], [PP10, 2.3]). This means that the functor of Fano manifolds in general is not separated. Nevertheless, even without knowing the

uniqueness we can still ask generally whether a ‘nice’ degeneration exists. Of course, this depends on the meaning of ‘nice.’ In this paper, we are looking for degenerations satisfying two properties.

First, the degenerate fibers should be mildly singular and still with negative canonical classes. Recall that a variety  $X$  is called a  $\mathbb{Q}$ -Fano variety if it only has klt singularities (see [KM98] for the meaning of the terminology) and  $-K_X$  is ample. In particular, a  $\mathbb{Q}$ -Fano variety is normal. This class of varieties plays a central role in birational geometry. From many viewpoints, it has a similar behavior as Fano manifolds.

*Definition 1.* Let  $f : (\mathcal{X}, \mathcal{L}) \rightarrow C$  be a flat projective morphism over a smooth curve, where  $\mathcal{X}$  is normal and  $\mathcal{L}$  is an  $f$ -ample  $\mathbb{Q}$ -line bundle. We call  $(\mathcal{X}, \mathcal{L}) \rightarrow C$  a polarized generic  $\mathbb{Q}$ -Fano family if there exists a nonempty open set  $C^* \subset C$  such that for any  $t \in C^*$ ,  $\mathcal{X}_t$  is klt and  $\mathcal{L}|_{\mathcal{X}^*} \sim_{\mathbb{Q}, C^*} -K_{\mathcal{X}^*}$ . In this case, we say  $C^*$  parametrizes nondegenerate fibers. If we can choose  $C^* = C$ , then we call  $f : (\mathcal{X}, -K_{\mathcal{X}/C}) \rightarrow C$  a  $\mathbb{Q}$ -Fano family,

We remark that a family being  $\mathbb{Q}$ -Fano is a more restrictive condition than being flat with every fiber  $\mathbb{Q}$ -Fano, because we also need that the canonical divisor of the global family is  $\mathbb{Q}$ -Cartier. Given a polarized generic  $\mathbb{Q}$ -Fano family  $(\mathcal{X}, \mathcal{L}) \rightarrow C$ , there exists a maximal open set  $C^* \subset C$  parametrizing nondegenerate fibers. Conversely, given a  $\mathbb{Q}$ -Fano family  $\mathcal{X}^*$  over  $C^*$  and  $C^* \subset C$  where  $C$  is a smooth curve, using the properness of the Hilbert scheme, we easily see  $\mathcal{X}^*$  can be extended to be a generic  $\mathbb{Q}$ -Fano family over  $C$ .

Second, we want our total family to minimize the following invariant, which is motivated by the intersection number interpretation (see [Wan12, Oda13a] or Section 8.1) of the original DF invariant defined for a test configuration (see Definitions 3 and 5). We refer to [Fut83], [FS90], [DT92], [Tia97], [Don02], [PT], [PT09] and Remark 1 in Section 7 for background and history about this important invariant.

*Definition 2* (Donaldson-Futaki invariant). Let  $\mathcal{L}$  be a relative nef  $\mathbb{R}$ -line bundle on  $\mathcal{X}$  over a proper smooth curve  $C$ . We assume that there is a nonempty open set  $C^* \subset C$ , such that for any  $t \in C^*$ ,  $\mathcal{X}_t$  is klt and  $\mathcal{L}|_{\mathcal{X}^*} \sim_{\mathbb{R}, C^*} -rK_{\mathcal{X}^*}$ , which is ample over  $C^*$ . We define the Donaldson-Futaki invariant (or DF invariant) to be

$$(1) \quad \text{DF}(\mathcal{X}/C, \mathcal{L}) = \frac{1}{2(n+1)(-K_{\mathcal{X}_t})^n} \left( (n+1)K_{\mathcal{X}/C} \cdot \left(\frac{1}{r}\mathcal{L}\right)^n + \left(\frac{1}{r}\mathcal{L}\right)^{n+1} \right).$$

This is proportional to the CM degree, which is the degree of the CM line bundle (see, e.g., [PT09]). It is a very important invariant for studying the family of Fano varieties whose positivity has not been fully understood. We refer Section 7 for more background, especially its relation with the existence of Kähler-Einstein metric.

Now we can state our main theorem, which roughly says a certain ‘nice’ degeneration exists.

**THEOREM 1.** *Assume that  $(\mathcal{X}, \mathcal{L}) \rightarrow C$  is a polarized generic  $\mathbb{Q}$ -Fano family over a smooth proper curve  $C$ . Let  $C^* \subset C$  parametrize nondegenerate fibers. We can construct a finite morphism  $\phi : C' \rightarrow C$  from a nonsingular curve  $C'$ , a  $\mathbb{Q}$ -Fano family  $(\mathcal{X}^s, \mathcal{L}^s := -K_{\mathcal{X}^s}) \rightarrow C'$  and a birational map  $\mathcal{X}^s \dashrightarrow \mathcal{X} \times_C C'$  that induces an isomorphism between the restrictions*

$$(\mathcal{X}^s, \mathcal{L}^s)|_{\phi^{-1}(C^*)} \cong (\mathcal{X}, \mathcal{L})|_{C^*} \times_C C'$$

such that

$$\mathrm{DF}(\mathcal{X}^s/C', -K_{\mathcal{X}^s}) \leq \mathrm{deg}(\phi) \cdot \mathrm{DF}(\mathcal{X}/C, \mathcal{L}),$$

and the equality holds for the construction only if  $(\mathcal{X}, \mathcal{L}) \rightarrow C$  is a  $\mathbb{Q}$ -Fano family.

**1.2. Outline of the proof.** In this subsection, we explain our strategy. The main idea of showing Theorem 1 is to modify a given a polarized generic  $\mathbb{Q}$ -Fano family and then to use the intersection formula to analyze the variation of Donaldson-Futaki invariants under modifications of the test configurations.

When the polarized generic  $\mathbb{Q}$ -Fano family arises from a test configuration, the authors in [RT07] and [ADVLN12] have also studied how to simplify a given test configuration. In particular, by using the (equivariant) semistable reduction theorem from [KKMSD73], it was shown (cf. [ADVLN12]) that any  $K$ -unstable polarized variety  $(X, L)$  can be destabilized by a test configuration whose central fibre  $\mathcal{X}_0^{\mathrm{tc}}$  is (reduced) simple normal crossing. In our article, we will apply the machinery of minimal model program with scaling to modify this semistable test configuration. As far as we know, the idea of applying the modern birational geometry to study  $K$ -stability algebraically is first contained in Odaka’s work (see [Oda13b]). Here we carry out a more refined study of the change of Donaldson-Futaki invariants under various birational modifications coming from MMP when  $X$  is Fano.

Our first observation is that the DF invariants of the polarizations always *decrease* along the direction of the canonical class of the test configuration. Of course, when we deform the polarization along the canonical class for a long enough time, we may hit the boundary of ample cone. Then MMP allows us to change the model and enables us to continue the deformation. So as long as it is before the pseudo-effective threshold, we still get polarizations but on some new models. In fact in birational geometry, this process is precisely the minimal model program with scaling, which is a central theme in many recent works; see, e.g., [BCHM10].

On the other hand, we can show if we run an MMP from the semi-stable model with the scaling of a suitably perturbed polarization, at the pseudo-effective threshold point we contract all but one components and end with a special test configuration. Since, by our observation, the Donaldson-Futaki invariants decrease along the sequence of MMP with scaling, this indeed finishes the proof of the K-semistable case.

However, to prove the K-stable case, we have to eliminate the perturbation term appearing when we perturb the polarization. This is more involved and therefore we have to take a more delicate process. In fact we have to divide the modification process into three steps.

*Step 1.* We first apply semi-stable reduction and run a relative MMP of this semi-stable family over  $\mathcal{X}$ . Then we achieve a model  $\mathcal{X}^{\text{lc}}$  that is the *log canonical modification* of the base change of  $(\mathcal{X}, \mathcal{X}_0)$ . We define the polarization  $\mathcal{L}^{\text{lc}}$  on  $\mathcal{X}^{\text{lc}}$  to be the sum of the pull back of the original polarization and a small positive multiple of  $K_{\mathcal{X}^{\text{lc}}}$ ; i.e., the new polarization on  $K_{\mathcal{X}^{\text{lc}}}$  is obtained by deforming the original one along the direction of the canonical class. Thus by our observation, we can check that the DF-invariants decrease. We note that the idea of running the MMP by passing through the log canonical modification is inspired by Odaka's work (see [Oda13a]). But here we only need to compute the derivative of the DF invariants. In Odaka's work [Oda13a], as he was studying the K-stability problem for general polarized varieties, a subtler estimate involving in terms of higher order was needed.

**THEOREM 2.** *Let  $(\mathcal{X}, \mathcal{L}) \rightarrow C$  be a polarized generic  $\mathbb{Q}$ -Fano family over a proper smooth curve, with  $C^* \subset C$  parametrizing the nondegenerate fibers. Then we can construct a finite morphism  $\phi : C' \rightarrow C$  and a polarized generic  $\mathbb{Q}$ -Fano family  $(\mathcal{X}^{\text{lc}}, \mathcal{L}^{\text{lc}}) \rightarrow C'$  with the following properties:*

- (1) *There is a birational morphism  $\mathcal{X}^{\text{lc}} \rightarrow \mathcal{X} \times_C C'$ , which is isomorphic over  $\phi^{-1}(C^*)$ .*
- (2) *For every  $t \in C'$ ,  $(\mathcal{X}^{\text{lc}}, \mathcal{X}_t^{\text{lc}})$  is log canonical.*
- (3) *There is an inequality*

$$\text{DF}(\mathcal{X}^{\text{lc}}/C', \mathcal{L}^{\text{lc}}) \leq \text{deg}(\phi) \cdot \text{DF}(\mathcal{X}/C, \mathcal{L}).$$

*Moreover, the equality holds for our construction if and only if  $(\mathcal{X}, \mathcal{X}_t)$  is log canonical for every  $t \in C$ , in which case  $\mathcal{X}^{\text{lc}}$  is isomorphic to  $\mathcal{X} \times_C C'$ .*

*Step 2.* Next, we will run the minimal model program with scaling. By letting  $l > 0$  be a sufficiently large integer, we can assume that  $\mathcal{H}^{\text{lc}} = \mathcal{L}^{\text{lc}} - (l+1)^{-1}(K_{\mathcal{X}^{\text{lc}}} + \mathcal{L}^{\text{lc}})$  is ample. Thus we can run  $K_{\mathcal{X}^{\text{lc}}}$ -MMP with scaling of  $\mathcal{H}^{\text{lc}}$  over  $C$  starting from the polarization  $K_{\mathcal{X}^{\text{lc}}} + (l+1)\mathcal{H}^{\text{lc}} = l\mathcal{L}^{\text{lc}}$ . It follows from [BCHM10] that the sequence of  $K_{\mathcal{X}^{\text{lc}}}$ -MMP with scaling of  $\mathcal{H}^{\text{lc}}$  over  $C$  will decrease the scaling factor until the pseudo-effective threshold is reached.

We denote its anti-canonical model by  $\mathcal{X}^{\text{ac}}$ . Since this is again deforming the polarization along the direction of the canonical class, we can also check that the DF invariants are continuously decreasing when we scale down the scaling factor. So we have the following theorem.

**THEOREM 3.** *Let  $(\mathcal{X}^{\text{lc}}, \mathcal{L}^{\text{lc}}) \rightarrow C$  be a polarized generic  $\mathbb{Q}$ -Fano family over a proper smooth curve, with  $C^* \subset C$  parametrizing the nondegenerate fibers. We assume that  $(\mathcal{X}^{\text{lc}}, \mathcal{X}_t^{\text{lc}})$  is log canonical for any  $t \in C$ . Then we can construct a polarized generic  $\mathbb{Q}$ -Fano family  $(\mathcal{X}^{\text{ac}}, \mathcal{L}^{\text{ac}}) \rightarrow C$  that is isomorphic to  $(\mathcal{X}^{\text{lc}}, \mathcal{L}^{\text{lc}})$  over  $C^*$  such that  $-K_{\mathcal{X}^{\text{ac}}} \sim_{\mathbb{Q}, C} \mathcal{L}^{\text{ac}}$ ,  $(\mathcal{X}^{\text{ac}}, \mathcal{X}_t^{\text{ac}})$  is log canonical for any  $t \in C$  and*

$$\text{DF}(\mathcal{X}^{\text{ac}}/C, \mathcal{L}^{\text{ac}}) \leq \text{DF}(\mathcal{X}^{\text{lc}}/C, \mathcal{L}^{\text{lc}}),$$

with the equality holding for our construction if and only if  $(\mathcal{X}^{\text{ac}}, \mathcal{L}^{\text{ac}}) \cong (\mathcal{X}^{\text{lc}}, \mathcal{L}^{\text{lc}})$ .

*Step 3.* At this point, MMP of  $\mathcal{X}^{\text{ac}}$  with scaling of  $\mathcal{L}^{\text{ac}} \sim_{\mathbb{Q}, C} -K_{\mathcal{X}^{\text{ac}}}$  will not produce any new model. Instead of running MMP, now we apply the technique of extending  $\mathbb{Q}$ -Fano varieties. So after possibly a base change, we obtain a  $\mathbb{Q}$ -Fano family  $\mathcal{X}^{\text{s}}$  that is isomorphic over the base that parametrizes nondegenerate fibers. Furthermore, from our construction of  $\mathbb{Q}$ -Fano extension, we know that  $\mathcal{X}_t^{\text{s}}$  is a divisor whose discrepancy with respect to  $\mathcal{X}^{\text{ac}}$  is 0.

Thus we can extract  $\mathcal{X}_t^{\text{s}}$  on  $\mathcal{X}^{\text{ac}}$  to get a model  $\mathcal{X}'$  such that  $-K_{\mathcal{X}'}$  is relatively big and nef, and  $\mathcal{X}' \dashrightarrow \mathcal{X}^{\text{s}}$  is a birational contraction. We observe that, over those models whose anti-canonical class is proportional to the polarization over the base, the DF invariant is just a linear function with negative leading coefficient on the volume of the anti-canonical class. Thus we easily see it decreases from  $\mathcal{X}'$  to  $\mathcal{X}^{\text{s}}$ .

**THEOREM 4.** *Let  $(\mathcal{X}^{\text{ac}}, \mathcal{L}^{\text{ac}}) \rightarrow C$  be a polarized generic  $\mathbb{Q}$ -Fano family over a proper smooth curve, where  $C^* \subset C$  parametrizes nondegenerate fibers. We assume that  $(\mathcal{X}^{\text{ac}}, \mathcal{X}_t^{\text{ac}})$  is log canonical for any  $t \in C$  and  $-K_{\mathcal{X}^{\text{ac}}} \sim_{\mathbb{Q}, C} \mathcal{L}^{\text{ac}}$ . Then after a finite base change  $\phi: C' \rightarrow C$ , we can construct a  $\mathbb{Q}$ -Fano family  $(\mathcal{X}^{\text{s}}, -K_{\mathcal{X}^{\text{s}}}) \rightarrow C'$ , which is isomorphic to  $(\mathcal{X}^{\text{ac}}, \mathcal{L}^{\text{ac}}) \times_C C'$  over  $\phi^{-1}(C^*)$ , such that*

$$\text{DF}(\mathcal{X}^{\text{s}}/C', -K_{\mathcal{X}^{\text{s}}}) \leq \deg(\phi) \cdot \text{DF}(\mathcal{X}^{\text{ac}}/C, -K_{\mathcal{X}^{\text{ac}}}),$$

and the equality holds for our construction if and only if  $(\mathcal{X}^{\text{s}}, -K_{\mathcal{X}^{\text{s}}})$  is a  $\mathbb{Q}$ -Fano family.

Finally, we outline the organization of Part 1. In Section 2, we review the results that we need from birational geometry and MMP. At the end we prove Theorem 6, which is a weaker form of Theorem 1 and will be needed for the proof of the general case. In Section 3, 4 and 5, we prove Theorem 2, 3 and 4 respectively. In Section 6, we put them together to obtain Theorem 1 the main theorem.

## 2. MMP and birational models

In this section, we give a few definitions as well as briefly introduce some preliminary results, especially the results in the minimal model program. Then we prove Theorem 6, which can be considered as a weaker version of Theorem 1. Later this form is needed for the purpose of calculating the variance of Donaldson-Futaki invariants and thus to address the main theorem.

*2.1. Notation and Conventions.* We follow [KM98] for the standard terminologies of birational geometry. In particular, see [KM98, 0.4] for the definitions of general concepts and [KM98, 2.34, 2.37] for the definitions of *klt*, *lc* and *dlt* singularities.

A smooth variety  $Y$  that is flat over a smooth curve  $C$  is called a *semi-stable family* if for any  $t \in C$ ,  $(Y, Y_t)$  is simple normal crossing.

Assume that  $X$  is proper over  $S$ . For any  $\mathbb{Q}$ -divisor  $D$  on  $X$ , we denote  $\bigoplus_m f_*(\mathcal{O}_X(\lfloor mD \rfloor))$  by  $R(X/S, D)$ . A  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor  $L$  on  $X$  is called *relatively semi-ample* if for sufficiently divisible  $m > 0$ ,  $f^*f_*\mathcal{O}_X(mL) \rightarrow \mathcal{O}_X(mL)$  is surjective.

Assume that  $f : (X, \Delta) \rightarrow S$  is a log canonical pair projective over  $S$ , where  $K_X + \Delta$  is big and semi-ample over  $S$ . Then we know  $R(X/S, K_X + \Delta)$  is a finitely generated  $\mathcal{O}_S$ -algebra, and we define

$$Y = \text{Proj } R(X/S, K_X + \Delta)$$

as *the relative log canonical model* of  $(X, \Delta)$  over  $S$ . We say  $X^m$  is a *good minimal model* of  $(X, \Delta)$  over  $S$  if  $h : X \dashrightarrow X^m$  is a minimal model of  $(X, \Delta)$  over  $S$  and  $K_{X^m} + h_*\Delta$  is relatively semi-ample.

Let  $(X, \Delta)$  be a normal pair; i.e.,  $X$  is a normal variety and  $\Delta = \sum a_i \Delta_i$  is a  $\mathbb{Q}$ -divisor with distinct prime divisors  $\Delta_i$  and rational numbers  $a_i$ . Assume  $0 \leq a_i \leq 1$ . We say that a birational projective morphism  $f : Y \rightarrow (X, \Delta)$  gives *the log canonical modification* of  $(X, \Delta)$  if with the divisor  $\Delta_Y = f_*^{-1}(\Delta) + E^{\text{lc}}$  on  $Y$ , the pair  $(Y, \Delta_Y)$  satisfies

- (1)  $(Y, \Delta_Y)$  is log canonical, and
- (2)  $K_Y + \Delta_Y$  is ample over  $X$ .

Here  $E^{\text{lc}}$  denotes the sum of  $f$ -exceptional prime divisors with coefficients 1.

Let  $X \rightarrow Y$  be a dominant morphism between normal varieties. A prime divisor  $E$  of  $X$  is called *horizontal* if  $E$  dominates  $Y$ ; otherwise it is called *vertical*. Given any divisor  $E$ , we can uniquely write  $E = E^v + E^h$ , where the horizontal part  $E^h$  consists of all the horizontal components and the vertical part  $E^v$  consists of all the vertical components.

For a model  $\bullet$  (e.g.,  $\mathcal{X}$ ) over  $C$  and  $0 \in C$  a point, we denote by  $\bullet_0$  its fiber over 0.

2.2. *MMP with scaling.* In this subsection, we briefly introduce the concept of MMP with scaling and summarize the results that we will need later. For more details see, e.g., [BCHM10].

Let  $(X, \Delta)/S$  be a klt pair that is projective over  $S$ . Let  $H$  be an ample class on  $X$ . Let  $\lambda \geq 0$  be a positive number such that  $K_X + \Delta + \lambda H$  is nef over  $S$ . For instance, we can take  $\lambda \gg 0$ . Denote  $(X^0, \Delta^0) := (X, \Delta)$  and  $\lambda_0 = \lambda$ .

Suppose we have defined a klt pair  $(X^i, \Delta^i)$  that is projective over  $S$ , a  $\mathbb{Q}$ -divisor  $H^i$  on  $X^i$ , and a positive number  $\lambda_i$  such that  $K_{X^i} + \Delta^i + \lambda_i H^i$  is nef over  $S$ . We first define

$$(2) \quad \lambda_{i+1} = \min\{\lambda | K_{X^i} + \Delta^i + \lambda H^i \text{ is nef over } S\}.$$

If  $\lambda_{i+1} = 0$ , then we stop. Otherwise, suppose we can choose a  $(K_{X^i} + \Delta^i)$ -negative extremal ray  $R \subset \text{NE}(X^i/S)$ , with  $R \cdot (K_{X^i} + \Delta^i + \lambda_{i+1} H^i) = 0$ . Assume that the extremal contraction induced by  $R$  is birational. Then we let  $X^{i+1}$  be the variety obtained by the  $(K_{X^i} + \Delta^i)$ -flip or divisorial contraction over  $S$  with respect to the curve class  $R$  (cf. [KM98, §3.7]) and  $\Delta^{i+1}$  (resp.  $H^{i+1}$ ) the push-forward of  $\Delta^i$  (resp.  $H^i$ ) to  $X^{i+1}$ . It is obvious that  $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_i \geq \dots$ . We call this sequence

$$(X^0, \Delta^0) \dashrightarrow (X^1, \Delta^1) \dashrightarrow \dots \dashrightarrow (X^i, \Delta^i) \dashrightarrow \dots$$

a sequence of  $(K_X + \Delta)$ -MMP with scaling of  $H$ ,  $\lambda$  the scaling factor and  $H$  the scaling divisor class (or the scaling divisor for abbreviation).

From the definition, we know that for any  $t \in [\lambda_{i+1}, \lambda_i]$ ,  $K_{X^i} + \Delta^i + tH^i$  is nef over  $S$ . Moreover, if  $t \in [\lambda_{i+1}, \lambda_i]$ , then  $X^i$  is a relatively minimal model of  $(X, \Delta + tH)$  over  $S$  because for  $j \leq i$ , the step  $X^{j-1} \dashrightarrow X^j$  of  $(K_X + \Delta)$ -MMP is also a step for  $(K_X + \Delta + tH)$ -MMP.

We need the following results.

**THEOREM 5.** *With the above notation, let  $(X, \Delta)$  be a klt pair that is projective over  $S$ . There exists a finite number  $i$  such that*

- (1) *if  $\Delta$  is big and  $K_X + \Delta$  is pseudo-effective over  $S$ , then after  $i$  steps,  $\lambda_i = 0$  and  $K_{X^i} + \Delta^i$  is semi-ample over  $S$ ;*
- (2) *if  $K_X + \Delta$  is not pseudo-effective, then after  $i$  steps,  $\lambda_i > 0$  is equal to the pseudo-effective threshold (see [BCHM10, 1.1.6] for the definition) of  $K_X + \Delta$  with respect to  $H$ , which is a rational number. Furthermore, if we let  $i$  be the smallest index such that  $\lambda_i$  is equal to the pseudo-effective threshold, then  $X^{i-1}$  is a good minimal model of  $(X, \Delta + \lambda_i H)$  over  $C$ .*

*Proof.* In (2), running  $(K_X + \Delta)$ -MMP with scaling of  $H$  is the same as running  $(K_X + \Delta + (\lambda_i - \varepsilon)H)$ -MMP with scaling of  $H$  for  $0 < \varepsilon < \lambda_i$ . Therefore these statements follow from [BCHM10, 1.1.7, 1.3.3 and 1.4.2].  $\square$

PROPOSITION 1. *Let  $(Y, \Delta_Y)$  be a klt pair projective over a smooth curve  $C$  with a relative ample class  $L$ . We assume that we can write  $K_Y + \Delta_Y + L \sim_{\mathbb{Q}, C} E = E^h + E^v \geq 0$  such that the horizontal part  $E^h$  is exceptional for a birational morphism  $Y \rightarrow X$ , and the vertical part  $E^v$  can be written as  $\sum a_i E_i^v$  where  $a_i > 0$ , and  $\text{Supp}(\sum E_i^v)$  does not contain any fiber. Then we have the following:*

- (1) *The  $(K_Y + \Delta_Y + L)$ -MMP with scaling of  $L$  will yield a model  $Y^i$  such that  $K_{Y^i} + \Delta_{Y^i} + L^i \sim_{\mathbb{Q}} 0$ , where  $\Delta_{Y^i}$  and  $L^i$  are the push forward of  $\Delta_Y$  and  $L$  on  $Y^i$ .*
- (2) *The divisors contracted by  $Y \dashrightarrow Y^i$  are precisely  $\text{Supp}(E)$ , and  $L^i$  on  $Y^i$  is relatively big and nef over  $C$ .*

*Proof.* From the assumption, we know that the pseudo-effective threshold of  $(Y, \Delta_Y)/C$  with respect to  $L$  is 1. Then by Theorem 5(2), this sequence of MMP yields a good minimal model  $Y^i$  of  $K_Y + \Delta_Y + L$  over  $C$ . Since  $K_{Y^i} + \Delta_{Y^i} + L^i$  is semi-ample, the map  $Y \dashrightarrow Y^i$  contracts precisely the divisorial component in  $\mathbf{B}(K_Y + \Delta_Y + L/C)$ , which is  $\text{Supp}(E)$ . In fact, it is easy to see this for the components in  $E^h$  since they are exceptional for a birational morphism. For  $E^v$ , by our assumption, it is of the insufficient fiber type (cf. [Lai11, 2.9]), so by [Lai11, 2.10] it is contained in

$$\mathbf{B}_-(K_Y + \Delta_Y + L/C) \subset \mathbf{B}(K_Y + \Delta_Y + L/C).$$

From the definition of MMP with scaling, we see that for any  $t \in [\lambda_{i+1}, \lambda_i]$ ,  $K_{Y^i} + \Delta_{Y^i} + tL^i$  is nef. Since  $K_{Y^i} + \Delta_{Y^i} + L^i \sim_{\mathbb{Q}, C} 0$  and by our assumption  $\lambda_i > \lambda_{i+1} = 1$ , then  $L^i$  is nef over  $C$ .  $\square$

2.3. *Log canonical modification and  $\mathbb{Q}$ -Fano extension.* Let  $f^* : \mathcal{X}^* \rightarrow C^*$  be a flat projective morphism,  $\mathcal{X}^*$  a klt variety and  $C^*$  the germ of a smooth curve. Let  $C$  be a smooth curve such that  $C^* = C \setminus \{0\}$ . Let  $\mathcal{X}$  be a normal compactification of  $\mathcal{X}^*$  that is projective over  $C$  such that  $\mathcal{X}^* = \mathcal{X} \times_C C^*$ . We first show a general result of the existence of the log canonical modification for the variety fibered over a curve. In fact, the log canonical modification is known to exist under more general assumptions (see, e.g., [OX12]). Here we just give a proof of the case that we need for the reader's convenience.

PROPOSITION 2. *With the above notation, assume that  $(\mathcal{X}, \mathcal{X}_0)$  admits a log resolution  $\mathcal{Y}$  such that  $\mathcal{Y}_0$  is reduced simple normal crossing. Then the log canonical modification  $\mathcal{X}^{\text{lc}} \rightarrow (\mathcal{X}, \mathcal{X}_0)$  exists and satisfies  $(\mathcal{X}^{\text{lc}}, \mathcal{X}_0^{\text{lc}})$  is log canonical.*

*Proof.* Let  $\pi : \mathcal{Y} \rightarrow (\mathcal{X}, \mathcal{X}_0)$  be a log resolution. If we write  $B$  to be the reduced divisor  $\text{Ex}(\pi)$ , it is well known that it suffices to show that  $(\mathcal{Y}, B + \pi_*^{-1} \mathcal{X}_0)$  has a relative log canonical model over  $\mathcal{X}$  (see [OX12, Lemma 2.2]).

Write

$$\pi^*K_{\mathcal{X}^*} + F^* = K_{\mathcal{Y}^*} + E^*,$$

where  $F^*$  and  $E^*$  are effective and without common components. Let  $E$  be the closure of  $E^*$  in  $\mathcal{Y}$ . Now we consider the pair  $(\mathcal{Y}, E + \delta G)$ , where  $G$  is the sum of the  $\pi$ -exceptional divisors whose centers are over  $C^*$  and  $0 < \delta \ll 1$  such that  $(\mathcal{Y}, E + \delta G)$  is klt. Then it follows from [BCHM10] that  $R(\mathcal{Y}/\mathcal{X}, K_{\mathcal{Y}} + E + \delta G)$  is a finitely generated  $\mathcal{O}_{\mathcal{X}}$ -algebra. By taking Proj, we obtain a model  $\phi: \mathcal{Y} \dashrightarrow \mathcal{X}^{\text{lc}}$  over  $\mathcal{X}$ . The model  $\mathcal{X}^{\text{lc}}$  is isomorphic to  $\text{Proj } R(\mathcal{Y}/\mathcal{X}, K_{\mathcal{Y}} + E + \delta G + \mathcal{Y}_0)$  since  $\mathcal{Y}_0$  is the pull back of a divisor from  $\mathcal{X}$ .

Because  $D = B + \pi_*^{-1}\mathcal{X}_0 - E - \delta G - \mathcal{Y}_0 \geq 0$  is an effective divisor, we know that

$$\begin{aligned} K_{\mathcal{Y}} + B + \pi_*^{-1}\mathcal{X}_0 - \phi^*\phi_*(K_{\mathcal{Y}} + E + \delta G + \mathcal{Y}_0) \\ \geq K_{\mathcal{Y}} + B + \pi_*^{-1}\mathcal{X}_0 - (K_{\mathcal{Y}} + E + \delta G + \mathcal{Y}_0) \geq 0. \end{aligned}$$

Since  $\phi$  contracts  $G$ , which is the same as  $\text{Supp}(D)$ , we easily see this implies that there is an isomorphism

$$R(\mathcal{Y}/\mathcal{X}, K_{\mathcal{Y}} + \pi_*^{-1}\mathcal{X}_0 + B) \cong R(\mathcal{Y}/\mathcal{X}, K_{\mathcal{Y}} + E + \delta G + \mathcal{Y}_0).$$

Hence we see  $\mathcal{X}^{\text{lc}}$  is indeed the log canonical modification of  $(\mathcal{X}, \mathcal{X}_0)$  and  $(\mathcal{X}^{\text{lc}}, \mathcal{X}_0^{\text{lc}})$  is log canonical as  $\phi_*(E + \delta G + \mathcal{Y}_0) = \mathcal{X}_0^{\text{lc}}$ .  $\square$

Next we study degenerations of Fano varieties.

*Example 1* (Degenerations of cubic surfaces). Let us consider a family of cubic surfaces

$$\mathcal{X} = (tf_3(x, y, z, w) + sxyz = 0) \subset \mathbb{P}(x, y, z, w) \times \mathbb{P}(s, t),$$

where  $f_3$  (resp.  $g_3$ ) is a general degree 3 homogeneous polynomial of  $x, y, z$  and  $w$  (resp.  $x, y$  and  $z$ ). Projecting to the second factor,  $\mathcal{X}$  are families of cubic surfaces over  $\mathbb{A}^1$  whose general fibers are smooth.

Now we modify  $\mathcal{X}$  in the following way: First we blow up the point

$$(0, 0, 0, 1) \in \mathcal{X}_0 = \sum_{i=1}^3 E_i = (xyz = 0) \subset \mathbb{P}^3$$

to get  $\mathcal{X}'$  and we denote the exceptional divisor by  $S_0 \cong \mathbb{P}^2$ . The fiber  $\mathcal{X}'_0$  has multiplicity 3 along  $S_0$ . Each birational transform of  $E_i$  is isomorphic to the  $\mathbb{P}^1$ -bundle  $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$  over  $\mathbb{P}^1$ .

Next we take a degree 3 base change  $\mathbb{P}[s_1, t_1] \rightarrow \mathbb{P}[s, t]$  that sends  $t$  (resp.  $s$ ) to  $t_1^3$  (resp.  $s_1^3$ ). Let  $\tilde{\mathcal{X}}$  be the normalization of  $\mathcal{X}' \times_{\mathbb{P}^1} \mathbb{P}^1$ . The pre-image  $S_1$  of  $S_0$  in  $\tilde{\mathcal{X}}$  is the degree 3 cover branched over the intersection of  $E$  and the birational transform of  $\sum_{i=1}^3 E_i$ , which is isomorphic to  $(xyz = 0) \subset \mathbb{P}(x, y, z) \cong S_0$ . Hence  $S_1$  is a cubic surface with three  $A_2$  singularities. We can contract the

preimage of the birational transform  $T$  of  $\sum_{i=1}^3 E_i$  and get a model  $\mathcal{X}^s$ . It is easy to see that  $\mathcal{X}_0^s \cong S_1$ .

This construction can indeed be generalized to arbitrary polarized generic  $\mathbb{Q}$ -Fano families in the following sense.

**THEOREM 6** ( $\mathbb{Q}$ -Fano extension). *Let  $f : \mathcal{X} \rightarrow C$  be a projective morphism over a smooth curve  $C$ . Assume that  $\mathcal{X}^* = \mathcal{X} \times_C C^*$  is a klt variety, where  $C^* = C \setminus \{0\}$ . We assume that there exists an ample  $\mathbb{Q}$ -divisor  $\mathcal{L}$  on  $\mathcal{X}$  such that  $\mathcal{L}|_{\mathcal{X}^*} \sim_{\mathbb{Q}, C^*} -K_{\mathcal{X}^*}$  and  $\mathcal{X}_t^*$  is a  $\mathbb{Q}$ -Fano variety for any  $t \in C^*$ .*

- (1) *There is a finite morphism  $\phi : C' \rightarrow C$  and a klt variety  $\mathcal{X}^s$  projective over  $C'$  such that the restriction of  $\mathcal{X}^s$  to the preimage  $\phi^{-1}(C^*)$  is isomorphic to  $\mathcal{X}^* \times_C C'$  and all fibers  $\mathcal{X}_t^s$  is  $\mathbb{Q}$ -Fano variety. In particular,  $\mathcal{X}_t^s$  is normal.*
- (2) *Moreover, if we assume that  $f^{\text{ac}} : \mathcal{X} = \mathcal{X}^{\text{ac}} \rightarrow C$  is a normal compactification such that  $\mathcal{L}^{\text{ac}} = -K_{\mathcal{X}^{\text{ac}}}$  is anti-ample and for any  $t \in C$ ,  $(\mathcal{X}_t^{\text{ac}}, \mathcal{X}_t^{\text{ac}})$  is log canonical. We can indeed get a  $\mathbb{Q}$ -Fano extension  $\mathcal{X}^s$  in (1) such that the divisor  $\mathcal{X}_t^s$  as a valuation over  $\tilde{\mathcal{X}}^{\text{ac}} := \mathcal{X} \times_C C'$  has the discrepancy  $a(\mathcal{X}_t^s; \tilde{\mathcal{X}}^{\text{ac}}) = 0$ .*

*Proof.* Let  $\phi : C' \rightarrow C$  be a finite base change such that  $\mathcal{X} \times_C C'$  yields a semi-stable log resolution  $\pi : \mathcal{Y} \rightarrow \mathcal{X} \times_C C'$ . We can assume that  $\mathcal{Y}$  yields an exceptional divisor  $A$  that is  $\pi$ -ample. Let  $\pi^*$  be the restriction of  $\pi$  over  $C^*$ . By abuse of notation, we will identify  $C = C'$  and  $\mathcal{X}^* = \mathcal{X} \times_C C'^*$ .

Write  $\pi^*(K_{\mathcal{X}^*}) + F^* = K_{\mathcal{Y}^*} + \Gamma^*$ , where  $\Gamma^*, F^* > 0$  are effective divisors without common components; by our assumption, the coefficients of  $\Gamma^*$  are less than 1. Let  $\Gamma, F$  be the closures of  $\Gamma^*$  and  $F^*$ . Let  $\varepsilon$  be a sufficiently small positive number such that  $\mathcal{L}_{\mathcal{Y}} := \pi^*\mathcal{L} + \varepsilon A$  is ample. We write  $A = A_1 + A_2$  such that  $A_2$  precisely consists of the vertical components that are over 0. By perturbing  $\mathcal{L}_{\mathcal{Y}}$  and reordering the components, we can assume that if we write

$$K_{\mathcal{Y}} + \mathcal{L}_{\mathcal{Y}} + \Gamma \sim_{\mathbb{Q}, C} (\varepsilon A_1 + F) + (\varepsilon A_2 + B)$$

with  $B$  supported on the fibers  $\mathcal{Y}_0 = \sum E_j$  and  $B + \varepsilon A_2 = \sum_{j=1}^k a_j E_j$ , then  $a_j > a_1$  if  $j > 1$ , where  $E_j$  ( $1 \leq j \leq k$ ) are all the components of  $\mathcal{Y}_0$ .

Let  $G$  be the sum of the prime divisors that are  $\pi$ -exceptional divisors whose centers are in  $\mathcal{X}^*$ . By choosing  $\varepsilon \ll \delta \ll 1$ , we assume that  $\delta G + \varepsilon A_1 \geq 0$  and its support is equal to  $G$ . We run  $(K_{\mathcal{Y}} + \mathcal{L}_{\mathcal{Y}} + \Gamma + \delta G)$ -MMP over  $\mathcal{X}$  with scaling of  $\mathcal{L}_{\mathcal{Y}}$ . By adding multiples of fibers, we can also assume that  $a_1 = 0$  in the above formula. Because

$$(K_{\mathcal{Y}} + \mathcal{L}_{\mathcal{Y}} + \Gamma + \delta G) \sim_{\mathbb{Q}, C} (\delta G + \varepsilon A_1 + F) + (\varepsilon A_2 + B - a_1 \mathcal{Y}_0) \geq 0,$$

whose support restricting on  $\mathcal{Y}^*$  contains all the exceptional divisors for  $\mathcal{Y}^* \rightarrow \mathcal{X}^*$ , we can apply Proposition 1 and conclude this sequence of MMP

terminates with a model  $\mathcal{X}^m$  satisfying that  $K_{\mathcal{X}^m} + \mathcal{L}^m \sim_{\mathbb{Q}, C} 0$ , the only remaining component over 0 is the birational transform of  $E_1$  and  $\mathcal{L}^m$  is big and nef. Thus we can define  $\mathcal{X}^s$  to be  $\text{Proj } R(\mathcal{X}^m/C, -K_{\mathcal{X}^m})$ . Over  $C^*$ , since  $\mathcal{Y}^* \dashrightarrow \mathcal{X}^{m*}$  contracts the same components as the ones of  $\mathcal{Y}^* \rightarrow \mathcal{X}^*$ , thus  $\mathcal{X}^{m*} (:= \mathcal{X}^m \times_C C^*) \dashrightarrow \mathcal{X}^*$  is isomorphic in codimension 1. Hence we see that

$$\mathcal{X}^* = \text{Proj } R(\mathcal{X}^*/C^*, \mathcal{L}|_{\mathcal{X}^*}) \cong \text{Proj } R(\mathcal{X}^{m*}/C^*, \mathcal{L}^m|_{\mathcal{X}^{m*}}) = \mathcal{X}^{s*}.$$

Representing  $\mathcal{L}_{\mathcal{Y}}$  by a general  $\mathbb{Q}$ -divisor, we can assume that  $(\mathcal{Y}, \Gamma + \delta G + \mathcal{Y}_0 + \mathcal{L}_{\mathcal{Y}})$  is dlt. The MMP sequence is also a sequence of  $(K_{\mathcal{Y}} + \Gamma + \delta G + \mathcal{Y}_0 + \mathcal{L}_{\mathcal{Y}})$ -MMP, thus  $(\mathcal{X}^m, \mathcal{X}_0^m + \mathcal{L}^m)$  is dlt. This implies that  $(\mathcal{X}^m, \mathcal{X}_0^m)$  is dlt since  $\mathcal{X}^m$  is  $\mathbb{Q}$ -factorial. As  $\mathcal{X}_0^m$  is irreducible, this indeed says  $(\mathcal{X}^m, \mathcal{X}_0^m)$  is plt, and so  $(\mathcal{X}^s, \mathcal{X}_0^s)$  is plt. By adjunction, we know  $\mathcal{X}_0^s$  is klt. This finishes the proof of (1).

For (2), we apply the same line of argument. We first choose  $\mathcal{X} = \mathcal{X}^{\text{ac}}$ . Then we know that we can write  $K_{\mathcal{Y}} + \Gamma = \pi^*(K_{\mathcal{X}^{\text{ac}}}) + F + B$ , where  $B$  is over  $\{0\}$ . Since  $(\mathcal{X}^{\text{ac}}, \mathcal{X}_0^{\text{ac}})$  is log canonical,  $\mathcal{X}^{\text{ac}}$  is canonical along  $\mathcal{X}_0^{\text{ac}}$ . So  $B \geq 0$ , whose support is the union of those divisors  $E_j \subset \mathcal{Y}_0$  such that  $a(E_j, \mathcal{X}^{\text{ac}}) > 0$ . Now we have

$$K_{\mathcal{Y}} + \mathcal{L}_{\mathcal{Y}} + \Gamma + \delta G \sim_{\mathbb{Q}, \mathcal{X}} B + F + \varepsilon A_1 + \varepsilon A_2 + \delta G,$$

whose vertical part over  $\{0\}$  is  $B + \varepsilon A_2$ . Thus by choosing  $0 < \varepsilon \ll 1$ , we can assume that after a small suitable perturbation, the divisor  $E_1$  having the smallest coefficient  $a_1$  is not contained in  $\text{Supp}(B)$ ; i.e., it satisfies that  $a(E_1, \mathcal{X}^{\text{ac}}) = 0$ . Then from the proof of (1),  $\mathcal{X}_0^m$  will be the birational image of such  $E_1$ .  $\square$

As investigated in [Kol07a], [HX09], the component  $E_1$  constructed in this way has many interesting geometric properties. Since any  $\mathbb{Q}$ -Fano variety is rationally connected (see [Zha06]), the argument that we just presented indeed gives a new proof of [HX09, Th. 3.1], which was originally obtained by applying Hacon-M<sup>c</sup>Kernan's extension theorem as in [HM07].

2.4. *The Zariski lemma.* Using this intersection formula, in the following work we need the higher dimensional analogue of the Zariski lemma for surfaces.

LEMMA 1. *Let  $\mathcal{X} \rightarrow C$  be a projective dominant morphism from an  $n$ -dimensional normal variety to a proper smooth curve. Let  $E$  be a  $\mathbb{Q}$ -divisor whose supports is contained in a fiber  $\mathcal{X}_0$ . Let  $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$  be  $n-2$  nef divisors on  $\mathcal{X}$ . Then*

$$E^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-2} \leq 0.$$

*If all  $\mathcal{L}_i$ 's are ample, then the equality holds if and only if  $E = t\mathcal{X}_0$  for some  $t \in \mathbb{Q}$ .*

*Proof.* When  $n = 2$ , this is the well-known Zariski lemma (see, e.g., [BHPVdV04, III.8.2]). We note that since  $E$  is not necessarily  $\mathbb{Q}$ -Cartier, here we have to use the intersection theory on normal surfaces developed by Mumford (see [Mum61]). For  $n > 2$ , as a nef bundle is the limit a sequence of ample  $\mathbb{Q}$ -line bundles, we can assume all  $\mathcal{L}_i$  are ample  $\mathbb{Q}$ -line bundles. Replacing  $\mathcal{L}_i$  by a multiple, we assume  $\mathcal{L}_i$  to be very ample. Adding a suitable multiple of the fiber, we can assume that  $E \geq 0$  and its support does not contain  $\text{Supp}(\mathcal{X}_0)$ . Therefore, we can cut  $\mathcal{X}$  by  $n - 2$  general sections in  $|\mathcal{L}_i|$  ( $1 \leq i \leq n - 2$ ) and reduce the question to the case when  $n = 2$ .  $\square$

### 3. Decreasing of DF invariant for the log canonical modification

Let  $(\mathcal{X}, \mathcal{L}) \rightarrow C$  be a polarized generic  $\mathbb{Q}$ -Fano family. It is easy to see all our operations will be local over  $C$ , so to simplify the notation, without loss of generality we will just denote one degenerate fiber to be  $\mathcal{X}_0$  and argue in a neighborhood of it.

In this section, we aim to verify Theorem 2. In fact, we will calculate the DF invariants on the log canonical modification  $\mathcal{X}^{\text{lc}} \rightarrow \mathcal{X}$ . We start from the line bundle  $\pi^{\text{lc}*}\mathcal{L}$  that is the pull back of the polarization on  $\mathcal{X}$ , whose DF invariant is equal to the original one. Since  $K_{\mathcal{X}^{\text{lc}}}$  is relative ample, if we deform  $\pi^{\text{lc}*}\mathcal{L}$  along the direction  $K_{\mathcal{X}^{\text{lc}}}$  sufficiently small amount, then we get an ample bundle on  $\mathcal{X}^{\text{lc}}$ . As this is a deformation along the canonical class, we can show the DF invariants decrease along this deformation.

**PROPOSITION 3.** *With the above notation, if  $\mathcal{X}^{\text{lc}}$  is not isomorphic to  $\mathcal{X}$ , then we can choose a polarization  $\mathcal{L}^{\text{lc}}$  on  $\mathcal{X}^{\text{lc}}$  such that*

$$\text{DF}(\mathcal{X}^{\text{lc}}/C, \mathcal{L}^{\text{lc}}) < \text{DF}(\mathcal{X}/C, \mathcal{L}).$$

*Proof.* By definition,  $K_{\mathcal{X}^{\text{lc}}}$  is  $\pi^{\text{lc}}$ -ample. We choose the relatively  $\pi^{\text{lc}}$ -ample  $\mathbb{Q}$ -divisor

$$E = K_{\mathcal{X}^{\text{lc}}} + \pi^{\text{lc}*}(\mathcal{L}).$$

Then  $E$  is  $\mathbb{Q}$ -linearly equivalent to a divisor whose support is contained in  $\mathcal{X}_0^{\text{lc}}$ . Since for sufficiently small rational  $\varepsilon$ ,  $\mathcal{L}_t^{\text{lc}} = \pi^{\text{lc}*}\mathcal{L} + tE$  is ample for any  $0 < t < \varepsilon$ , we see that  $(\mathcal{X}^{\text{lc}}, \mathcal{L}^{\text{lc}}) \rightarrow C$  is also a polarized generic  $\mathbb{Q}$ -Fano family.

Using the formula, we compute the derivative at  $t_0 \in (0, \varepsilon)$ :

$$\begin{aligned} \frac{d}{dt} \text{DF}(\mathcal{X}^{\text{lc}}/C, \mathcal{L}_t^{\text{lc}})|_{t_0} &= n(n+1)C_0 \cdot ((\mathcal{L}_{t_0}^{\text{lc}})^n \cdot E + K_{\mathcal{X}^{\text{lc}}} \cdot (\mathcal{L}_{t_0}^{\text{lc}})^{n-1} \cdot E) \\ &= C_1 \cdot (\mathcal{L}_{t_0}^{\text{lc}})^{n-1} \cdot E \cdot (\mathcal{L}_{t_0}^{\text{lc}} + K_{\mathcal{X}^{\text{lc}}}) = C_1 \cdot (\mathcal{L}_{t_0}^{\text{lc}})^{n-1} \cdot E^2, \end{aligned}$$

where  $C_0$  and  $C_1$  are positive numbers. By Lemma 1, the intersection  $(\mathcal{L}_{t_0}^{\text{lc}})^{n-1} \cdot E^2 \leq 0$  and it is zero if and only if  $E = K_{\mathcal{X}^{\text{lc}}} + \pi^{\text{lc}*}\mathcal{L}$  is  $\mathbb{Q}$ -linearly equivalent

to  $a\mathcal{X}_0^{\text{lc}}$  for some  $a$ . But this implies that  $\mathcal{X}^{\text{lc}} \cong \mathcal{X}$  since  $K_{\mathcal{X}^{\text{lc}}} \sim_{\mathbb{Q}, \mathcal{X}} E$  is  $\pi^{\text{lc}}$ -ample.  $\square$

*Proof of Theorem 2.* First we can take the base change  $\mathcal{X} \times_C C'$  such that its normalization  $\tilde{\mathcal{X}}$  admits a semi-stable reduction  $\mathcal{Y}$ . In particular,  $\tilde{\mathcal{X}}_0$  is reduced. Let  $\phi_{\mathcal{X}} : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  be the natural finite morphism, and let  $\tilde{\mathcal{L}} = \phi_{\mathcal{X}}^* \mathcal{L}$ . We first note that

CLAIM 1.  $\deg(C'/C) \cdot \text{DF}(\mathcal{X}/C, \mathcal{L}) \geq \text{DF}(\tilde{\mathcal{X}}/C', \tilde{\mathcal{L}})$ . Furthermore, the equality holds if and only if  $\mathcal{X}_0$  is reduced.

Indeed, by the pull-back formula for the log differential, we have  $K_{\tilde{\mathcal{X}}} + \tilde{\mathcal{X}}_0 = f^*(K_{\mathcal{X}} + \text{red}(\mathcal{X}_0))$  and  $K_{C'} + \{0'\} = \phi^*(K_C + \{0\})$ . So

$$K_{\tilde{\mathcal{X}}/C'} = f^*(K_{\mathcal{X}/C} + (\text{red}(\mathcal{X}_0) - \mathcal{X}_0)),$$

and the claim follows from the projection formula.

Now it follows from Proposition 2 that the log canonical modification  $\pi^{\text{lc}} : \mathcal{X}^{\text{lc}} \rightarrow \tilde{\mathcal{X}}$  exists and satisfies that  $\pi^{\text{lc}}$  is an isomorphism over  $\phi^{-1}(C^*)$ . Then Proposition 3 shows that

$$\text{DF}(\tilde{\mathcal{X}}/C', \tilde{\mathcal{L}}) \geq \text{DF}(\mathcal{X}^{\text{lc}}/C', \mathcal{L}^{\text{lc}}).$$

If  $(\mathcal{X}, \mathcal{X}_0)$  is log canonical, then  $\mathcal{X}_0$  is reduced and  $(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_0)$  is log canonical (cf. [KM98, 5.20]), which implies  $\mathcal{X}^{\text{lc}} \cong \tilde{\mathcal{X}}$ ; therefore the equality holds.

Conversely,  $\deg(\phi) \cdot \text{DF}(\mathcal{X}/C, \mathcal{L}) = \text{DF}(\mathcal{X}^{\text{lc}}/C', \mathcal{L}^{\text{lc}})$  is equivalent to saying the above two inequalities are indeed equalities. By Proposition 3 and the above claim, this holds only if  $\mathcal{X}_0$  is reduced and  $\mathcal{X}^{\text{lc}} \cong \tilde{\mathcal{X}}$ , which implies  $(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_0)$  is log canonical. Since

$$\phi_{\mathcal{X}}^*(K_{\mathcal{X}} + \mathcal{X}_0) = K_{\tilde{\mathcal{X}}} + \tilde{\mathcal{X}}_0,$$

it follows that  $(\mathcal{X}, \mathcal{X}_0)$  is also log canonical (see [KM98, 5.20]).  $\square$

#### 4. MMP with scaling

In this section, we aim to prove Theorem 3. We will apply the idea that the Donaldson-Futaki invariants decrease if we deform the polarization along the direction of the canonical class of the total family in ‘a long time’ process. To keep the deformed line bundle being a polarization, we have to do a sequence of surgeries on the family. In algebraic geometry, this surgery is given by the MMP with scaling. (See [BCHM10] and Section 2.2.)

4.1. *Running MMP.* By taking  $l > 0$  to be a sufficiently large integer, we can make  $\mathcal{H}^{\text{lc}} = \mathcal{L}^{\text{lc}} - (l+1)^{-1}(\mathcal{L}^{\text{lc}} + K_{\tilde{\mathcal{X}}}^{\text{lc}})$  ample. Let  $\lambda_0 = l+1$ . We let  $\mathcal{X}^0 = \mathcal{X}^{\text{lc}}$ ,  $\mathcal{L}^0 = \mathcal{L}^{\text{lc}}$  and  $\mathcal{H}^0 = \mathcal{H}^{\text{lc}}$ . Then  $K_{\mathcal{X}^0} + \lambda_0 \mathcal{H}^0 = l\mathcal{L}^{\text{lc}}$  is relatively ample.

Given an exceptional divisor  $E$ , if its center dominates  $C$ , then  $a(E, \mathcal{X}^0) > -1$  because  $\mathcal{X}^*$  is klt; if its center is vertical over  $C$ , then  $a(E, \mathcal{X}^0) \geq 0$  since  $(\mathcal{X}^0, \mathcal{X}_t^0)$  is log canonical for any  $t$  in  $C$ . In particular,  $\mathcal{X}^0$  is klt. To simplify the family, we run a sequence of  $K_{\mathcal{X}^0}$ -MMP over  $C$  with scaling of  $\mathcal{H}^0$  as in Section 2.2. So we obtain a sequence of models

$$\mathcal{X}^0 \dashrightarrow \mathcal{X}^1 \dashrightarrow \dots \dashrightarrow \mathcal{X}^k.$$

Recall that, as in Section 2.2, we have a sequence of critical value of scaling factors

$$\lambda_{i+1} = \min\{\lambda \mid K_{\mathcal{X}^i} + \lambda \mathcal{H}^i \text{ is nef over } C\},$$

with  $l+1 = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_k > \lambda_{k+1} = 1$ . Note that  $\lambda_{k+1} = 1$  is the pseudo-effective threshold of  $K_{\mathcal{X}^0}$  with respect to  $\mathcal{H}^0$  over  $C$  since it is the pseudo-effective threshold for the generic fiber. Any  $\mathcal{X}^i$  appearing in this sequence of  $K_{\mathcal{X}^0}$ -MMP with scaling of  $\mathcal{H}^0$  is a relative weak log canonical model of  $(\mathcal{X}^0, t\mathcal{H}^0)$  for any  $t \in [\lambda_i, \lambda_{i+1}]$ . (See [BCHM10, 3.6.7] for the definition of weak log canonical model.)

For  $\lambda > 1$ , we denote

$$(3) \quad \mathcal{L}_\lambda = \frac{1}{\lambda - 1} (K_{\mathcal{X}^0} + \lambda \mathcal{H}^0).$$

Let  $\mathcal{L}_\lambda^i$  (resp.  $\mathcal{H}^i$ ) be the push forward of  $\mathcal{L}^0$  (resp.  $\mathcal{H}^0$ ) to  $\mathcal{X}^i$ . As is clear from the context, this should not be confused with the  $i$ -th power or intersection product of  $\mathcal{L}_\lambda$  (resp.  $\mathcal{H}$ ). We note that by definition,  $\mathcal{L}_{\lambda+1}^0 = \mathcal{L}^0$ .

Given a  $\lambda$ , we will be interested in those  $i$  that satisfy the condition that  $\lambda_i \geq \lambda \geq \lambda_{i+1}$ . Note that

$$(4) \quad K_{\mathcal{X}^i} + \mathcal{L}_\lambda^i = \frac{\lambda}{\lambda - 1} (K_{\mathcal{X}^i} + \mathcal{H}^i).$$

LEMMA 2.  $-K_{\mathcal{X}^k} \sim_{\mathbb{Q}, C} \mathcal{L}_{\lambda_k}^k$  is big and semi-ample over  $C$ .

*Proof.* Since  $\lambda_k > \lambda_{k+1} = 1$ , by (4),

$$K_{\mathcal{X}^k} + \mathcal{L}_{\lambda_k}^k \sim_{\mathbb{Q}} \frac{\lambda_k}{\lambda_k - 1} (K_{\mathcal{X}^k} + \mathcal{H}^k).$$

This line bundle is relatively nef over  $C$ , and its restriction to the generic fiber is trivial, so it is  $\mathbb{Q}$ -linearly equivalent to a linear sum of components of  $\mathcal{X}_0^k$ . By its nefness, we can apply Lemma 1 to get

$$K_{\mathcal{X}^k} + \mathcal{L}_{\lambda_k}^k \sim_{\mathbb{Q}, C} 0.$$

By (3),  $\mathcal{L}_{\lambda_k}^k$  is proportional to  $K_{\mathcal{X}^k} + \lambda_k \mathcal{H}^k$ , which is big because  $\lambda_k > 1$ . From the relative base-point free theorem (cf. Theorem 3.3 in [KM98]), it is semi-ample over  $C$ .  $\square$

By the above lemma, we can define

$$\mathcal{X}^{\text{ac}} = \text{Proj } R(\mathcal{X}^k/C, \mathcal{L}_{\lambda_k}^k) = \text{Proj } R(\mathcal{X}^k/C, -K_{\mathcal{X}^k/C}).$$

Since  $(\mathcal{X}^0, \mathcal{X}_0^0)$  is log canonical and  $\mathcal{X}_0^0 = (f \circ \pi^{\text{lc}})^*(\{0\})$ , this is also a sequence of  $(K_{\mathcal{X}^0} + \mathcal{X}_0^0)$ -MMP, and thus  $(\mathcal{X}^k, \mathcal{X}_0^k)$  is log canonical, which implies that  $(\mathcal{X}^{\text{ac}}, \mathcal{X}_0^{\text{ac}})$  is log canonical as well.

4.2. *Decreasing of DF-invariant.* For any  $\lambda > 1$ , the restriction of  $K_{\mathcal{X}^0} + \lambda\mathcal{H}^0$  over  $C^*$  is relatively ample. So the MMP with scaling does not change  $\mathcal{X}^0 \times_C C^*$ ; i.e.,  $\mathcal{X}^0 \times_C C^* \cong \mathcal{X}^i \times_C C^*$  for any  $i \leq k$ .

Note that by the above lemma and projection formula,

$$\text{DF}(\mathcal{X}^k/C, \mathcal{L}_{\lambda_k}^k) = \text{DF}(\mathcal{X}^k/C, -K_{\mathcal{X}^k}) = \text{DF}(\mathcal{X}^{\text{ac}}/C, -K_{\mathcal{X}^{\text{ac}}}).$$

So Theorem 3 follows from the following proposition.

PROPOSITION 4. *With the notation as above, we have*

$$\text{DF}(\mathcal{X}^0/C, \mathcal{L}^0) \geq \text{DF}(\mathcal{X}^k/C, \mathcal{L}_{\lambda_k}^k) = \text{DF}(\mathcal{X}^k/C, -K_{\mathcal{X}^k}).$$

*The first equality holds if and only if  $h : \mathcal{X}^0 \dashrightarrow \mathcal{X}^k$  is an isomorphism.*

To prove Proposition 4, we first study how DF invariants change when we run MMP with scaling and modify the polarization correspondingly.

4.2.1. *Decreasing of DF on a fixed model.* Assume that  $\mathcal{X}_0^0 = \sum_{j \in I} E_j$ , where the  $E_j$ 's are the prime divisors. Since  $(\mathcal{X}^0, \mathcal{L}_\lambda^0) \times_C C^*$  is isomorphic to  $(\mathcal{X}^0 \times_C C^*, -K_{\mathcal{X}^0 \times_C C^*})$ , there exist  $a_j(\lambda) \in \mathbb{R}$  such that

$$K_{\mathcal{X}^0} + \mathcal{L}_\lambda^0 \sim_{\mathbb{R}, C} \sum_{j \in I} a_j(\lambda) E_j.$$

On  $\mathcal{X}^i$ , for any rational number  $\lambda > 1$  satisfying  $\lambda_i \geq \lambda \geq \lambda_{i+1}$ , we know  $\mathcal{L}_\lambda^i$  is big and semi-ample. Let  $\mathcal{Z}_\lambda$  be the relative log canonical model of  $(\mathcal{X}^0, \lambda\mathcal{H}^0)$  over  $C$ . Then there is a morphism  $\pi_\lambda : \mathcal{X}^i \rightarrow \mathcal{Z}_\lambda$  and a relatively ample  $\mathbb{Q}$ -divisor  $\mathcal{M}_\lambda$  on  $\mathcal{Z}_\lambda$  whose pull back is

$$\frac{1}{\lambda - 1} (K_{\mathcal{X}^i} + \lambda\mathcal{H}^i) = \mathcal{L}_\lambda^i.$$

LEMMA 3. *If  $\lambda_i \geq a > b \geq \lambda_{i+1}$  and  $b > 1$ , then  $\text{DF}(\mathcal{X}^i, \mathcal{L}_a^i) \geq \text{DF}(\mathcal{X}^i, \mathcal{L}_b^i)$ . The inequality is strict if there is a rational number  $\lambda \in [a, b]$  such that the push forward of  $\sum_{j \in I} a_j(\lambda) E_j$  to  $\mathcal{Z}_\lambda$  is not a multiple of the pull back of  $0 \in C$  on  $\mathcal{Z}_\lambda$ .*

*Proof.* We compute the derivative of the DF invariants in a way similar to Proposition 3:

$$\begin{aligned} \frac{d}{d\lambda} \text{DF}(\mathcal{X}^i/C, \mathcal{L}_\lambda^i) &= C_0 \left( (\mathcal{L}_\lambda^i)^{n-1} \cdot (\mathcal{L}_\lambda^i)' \cdot (\mathcal{L}_\lambda^i + K_{\mathcal{X}^i}) \right) \\ &= -\frac{C_0}{\lambda(\lambda-1)} (\mathcal{L}_\lambda^i)^{n-1} \cdot (K_{\mathcal{X}^i} + \mathcal{L}_\lambda^i)^2 \\ &= -\frac{C_0}{\lambda(\lambda-1)} (\mathcal{L}_\lambda^i)^{n-1} \cdot \left( \sum_{j \in I} a_j(\lambda) E_j \right)^2, \end{aligned}$$

where  $C_0$  is a positive constant. Then Lemma 3 follows from Lemma 1.  $\square$

4.2.2. *Invariance of DF at contraction or flip points.* If  $\lambda_{i+1} > 1$ , then by the definition of MMP with scaling, we pick up a  $K_{\mathcal{X}^i}$ -negative extremal ray  $[R]$  in  $\text{NE}(\mathcal{X}^i/C)$  such that  $R \cdot (K_{\mathcal{X}^i} + \lambda_{i+1} \mathcal{H}^i) = 0$ . We perform a birational transformation

$$\begin{array}{ccc} \mathcal{X}^i & \xrightarrow{f^i} & \mathcal{Y}^i \\ & \searrow & \swarrow \\ & \mathbb{P}^1 & \end{array}$$

which contracts all curves  $R'$  whose classes  $[R']$  are in the ray  $\mathbb{R}_{>0}[R]$ . There are two cases:

- (1) (Divisorial Contraction). If  $f^i$  is a divisorial contraction, then  $\mathcal{X}^{i+1} = \mathcal{Y}^i$ . Since  $f^i$  is a  $(K_{\mathcal{X}^i} + \lambda_{i+1} \mathcal{H}^i)$ -trivial morphism by the definition of the MMP with scaling, we have

$$K_{\mathcal{X}^i} + \lambda_{i+1} \mathcal{H}^i = (f^i)^*(K_{\mathcal{Y}^i} + \lambda_{i+1} \mathcal{H}^{i+1}),$$

which implies

$$\mathcal{L}_{\lambda_{i+1}}^i = (f^i)^* \mathcal{L}_{\lambda_{i+1}}^{i+1}.$$

Then it follows from Definition 2 and the projection formula that

$$\text{DF}(\mathcal{X}^i/C, \mathcal{L}_{\lambda_{i+1}}^i) = \text{DF}(\mathcal{X}^{i+1}/C, \mathcal{L}_{\lambda_{i+1}}^{i+1}).$$

- (2) (Flipping Contraction). If  $f^i$  is a flipping contraction, let  $\phi^i : \mathcal{X}^i \dashrightarrow \mathcal{X}^{i+1}$  be the flip:

$$\begin{array}{ccc} \mathcal{X}^i & \overset{\phi^i}{\dashrightarrow} & \mathcal{X}^{i+1} \\ & \searrow f^i & \swarrow f^{i+} \\ & \mathcal{Y}^i & \end{array}$$

$-K_{\mathcal{X}^i}$  is  $f^i$ -ample   $K_{\mathcal{X}^{i+1}}$  is  $f^{i+}$ -ample

As  $f^i$  is a  $K_{\mathcal{X}^i} + \lambda_{i+1} \mathcal{H}^i$ -trivial morphism,  $K_{\mathcal{X}^i} + \lambda_{i+1} \mathcal{H}^i = (f^i)^* D_{\mathcal{Y}^i}$  for some divisor  $D_{\mathcal{Y}^i}$ . Since  $f^i, f^{i+}, \phi^i$  are isomorphisms in codimension 1, we

also have  $K_{\mathcal{X}^{i+1}} + \lambda_{i+1}\mathcal{H}^{i+1} = (f^{i+1})^*D_{\mathcal{Y}^i}$ . Therefore, using the projection formula, we see that

$$\mathrm{DF}(\mathcal{X}^i/C, K_{\mathcal{X}^i} + \lambda_{i+1}\mathcal{H}^i) = \mathrm{DF}(\mathcal{Y}^i/C, D_{\mathcal{Y}^i}) = \mathrm{DF}(\mathcal{X}^{i+1}/C, K_{\mathcal{X}^{i+1}} + \lambda_{i+1}\mathcal{H}^{i+1}).$$

Now we can finish the proof of Proposition 4.

*Completion of proof of Proposition 4.* By the discussion in 4.2.1 and 4.2.2, we have

$$\begin{aligned} & \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}_{\lambda_0}^0) \geq \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}_{\lambda_1}^0) \\ & = \mathrm{DF}(\mathcal{X}^1/C, \mathcal{L}_{\lambda_1}^1) \geq \mathrm{DF}(\mathcal{X}^1/C, \mathcal{L}_{\lambda_2}^1) \\ & \quad \dots \quad \dots \quad \dots \\ & = \mathrm{DF}(\mathcal{X}^i/C, \mathcal{L}_{\lambda_i}^i) \geq \mathrm{DF}(\mathcal{X}^i/C, \mathcal{L}_{\lambda_{i+1}}^i) \\ & = \mathrm{DF}(\mathcal{X}^{i+1}/C, \mathcal{L}_{\lambda_{i+1}}^{i+1}) \geq \mathrm{DF}(\mathcal{X}^{i+1}/C, \mathcal{L}_{\lambda_{i+2}}^{i+1}) \\ & \quad \dots \quad \dots \quad \dots \\ & = \mathrm{DF}(\mathcal{X}^k/C, \mathcal{L}_{\lambda_k}^k) = \mathrm{DF}(\mathcal{X}^k/C, -K_{\mathcal{X}^k}). \end{aligned}$$

We proceed to characterize the equality case. Since  $-K_{\mathcal{X}^k} \sim_{\mathbb{Q}, C} \mathcal{L}_{\lambda_k}^k$  is relatively nef over  $C$ , we conclude that  $f^{k-1} : \mathcal{X}^{k-1} \rightarrow \mathcal{X}^k$  can only be a divisorial contraction. Therefore,  $h : \mathcal{X}^0 \dashrightarrow \mathcal{X}^k$  contracts at least one divisor if it is not an isomorphism.

Since  $\mathcal{X}^k$  is a minimal model of  $(\mathcal{X}^0, \mathcal{H}^0)$  (see Theorem 5(2)), we know that

$$0 < E = K_{\mathcal{X}^0} + \mathcal{H}^0 - h^*(K_{\mathcal{X}^k} + \mathcal{H}^k) \sim_{\mathbb{Q}, C} K_{\mathcal{X}^0} + \mathcal{H}^0,$$

which is supported on the fiber over 0. From the fact that the support of  $E$  is a proper subset of  $\mathcal{X}_0^0$ , it follows that  $K_{\mathcal{X}^0} + \mathcal{H}^0$  is not  $\mathbb{Q}$ -linearly equivalent to 0 over  $C$ ; i.e., the equality condition of Lemma 3 cannot hold on  $\mathcal{X}^{\mathrm{lc}}$ . Thus a for sufficiently small rational number  $\varepsilon$ ,

$$\mathrm{DF}(\mathcal{X}^{\mathrm{lc}}/C, \mathcal{L}^{\mathrm{lc}}) = \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}_{\lambda_0}^0) > \mathrm{DF}(\mathcal{X}^0/C, \mathcal{L}_{\lambda_0 - \varepsilon}^0) \geq \mathrm{DF}(\mathcal{X}^k/C, \mathcal{L}_{\lambda_k}^k). \quad \square$$

## 5. Revisiting of $\mathbb{Q}$ -Fano extension

Let us continue the study of Example 1.

*Example 2.* We use the notation of Example 1. Since  $K_{\mathcal{X}}$  is of bidegree  $(-1, -1)$ ,

$$\mathrm{DF}(\mathcal{X}/\mathbb{P}^1, -K_{\mathcal{X}}) = -\frac{1}{2(n+1)(-K_{\mathcal{X}_t})^n} (-K_{\mathcal{X}/\mathbb{P}^1})^{n+1} = \frac{4}{9}.$$

Using the intersection formula, we easily see

$$\begin{aligned}
\mathrm{DF}(\tilde{\mathcal{X}}/\mathbb{P}^1, -K_{\tilde{\mathcal{X}}}) &= \mathrm{DF}(\tilde{\mathcal{X}}/\mathbb{P}^1, -(K_{\tilde{\mathcal{X}}} + \tilde{\mathcal{X}}_0)) \\
&= 3 \cdot \mathrm{DF}(\mathcal{X}'/\mathbb{P}^1, -(K_{\mathcal{X}'} + \mathcal{X}'_0)) \\
&= 3 \cdot \mathrm{DF}(\mathcal{X}/\mathbb{P}^1, -(K_{\mathcal{X}} + \mathcal{X}_0)) \\
&= 3 \cdot \mathrm{DF}(\mathcal{X}/\mathbb{P}^1, -K_{\mathcal{X}}) \\
&= \frac{4}{3}.
\end{aligned}$$

Since  $K_{\tilde{\mathcal{X}}}|_{S_1} = (K_{\tilde{\mathcal{X}}} + \tilde{\mathcal{X}}_0)|_{S_1}$  is trivial, we calculate

$$\begin{aligned}
\mathrm{DF}(\mathcal{X}^s/\mathbb{P}^1, -K_{\mathcal{X}^s}) &= \frac{1}{18}(K_{\mathcal{X}^s/\mathbb{P}^1})^3 \\
&= \frac{1}{18}(K_{\tilde{\mathcal{X}}/\mathbb{P}^1} - T)^3 = \frac{1}{18}(K_{\tilde{\mathcal{X}}/\mathbb{P}^1} - \tilde{\mathcal{X}}_0 + S_1)^3 \\
&= \frac{4}{3} - \frac{1}{18}(9 - 3) < 3 \cdot \mathrm{DF}(\mathcal{X}/\mathbb{P}^1, -K_{\mathcal{X}}).
\end{aligned}$$

Therefore, we see that if we normalize the DF invariants by dividing the degree of the base change, then our process in this example decreases this normalized DF invariant.

From the discussion of the last section, we achieve a model  $\mathcal{X}^{\mathrm{ac}}$  over  $C$  with polarization  $\mathcal{L}^{\mathrm{ac}}$  that compactifies  $\mathcal{X}^*/C^*$  such that  $\mathcal{L}^{\mathrm{ac}} \sim_{\mathbb{Q}, C} -K_{\mathcal{X}^{\mathrm{ac}}}$  and  $(\mathcal{X}^{\mathrm{ac}}, \mathcal{X}_t^{\mathrm{ac}})$  is log canonical for any  $t \in C$ . We cannot run an MMP directly from  $\mathcal{X}^{\mathrm{ac}}$  to get  $\mathcal{X}^s$ . Instead we will resolve  $\mathcal{X}^{\mathrm{ac}}$  again and run MMP. More precisely, by Theorem 6(2), we know that there exists  $\phi : C' \rightarrow C$  with a  $\mathbb{Q}$ -Fano family  $\mathcal{X}^s/C'$ . We will show this is our final  $\mathbb{Q}$ -Fano family by verifying the decreasing of the DF invariant.

Using the notation in Theorem 6,  $a(\mathcal{X}_0^s; \tilde{\mathcal{X}}^{\mathrm{ac}}) = 0$  implies

$$a(\mathcal{X}_0^s; \tilde{\mathcal{X}}^{\mathrm{ac}}, \tilde{\mathcal{X}}_0^{\mathrm{ac}}) = -1$$

since  $\tilde{\mathcal{X}}_0^{\mathrm{ac}}$  is Cartier and  $(\tilde{\mathcal{X}}^{\mathrm{ac}}, \tilde{\mathcal{X}}_0^{\mathrm{ac}})$  is log canonical. Then for any number  $\lambda \in [0, 1]$ , we know that

$$a(\mathcal{X}_0^s; \tilde{\mathcal{X}}^{\mathrm{ac}}, \lambda \tilde{\mathcal{X}}_0^{\mathrm{ac}}) = -\lambda.$$

In particular, there exists a model  $\pi' : \mathcal{X}' \rightarrow \tilde{\mathcal{X}}^{\mathrm{ac}}$  that precisely extracts the divisor  $\mathcal{X}_0^s$  (cf. [BCHM10, 1.4.3]). Since  $a(\mathcal{X}_0^s; \tilde{\mathcal{X}}^{\mathrm{ac}}) = 0$ , we know  $\pi'^*(K_{\tilde{\mathcal{X}}^{\mathrm{ac}}}) = K_{\mathcal{X}'}$ . Then by the projection formula,

$$\mathrm{DF}(\mathcal{X}'/C', -K_{\mathcal{X}'}) = \mathrm{DF}(\tilde{\mathcal{X}}^{\mathrm{ac}}/C', -K_{\tilde{\mathcal{X}}^{\mathrm{ac}}}) = \deg(\phi) \cdot \mathrm{DF}(\mathcal{X}^{\mathrm{ac}}/C, -K_{\mathcal{X}^{\mathrm{ac}}}).$$

**PROPOSITION 5.** *We have the inequality*

$$\mathrm{DF}(\mathcal{X}'/C', -K_{\mathcal{X}'}) \geq \mathrm{DF}(\mathcal{X}^s/C', -K_{\mathcal{X}^s}),$$

and the equality holds if and only if the rational map  $\tilde{\mathcal{X}}^{\text{an}} \dashrightarrow \mathcal{X}^{\text{s}}$  is an isomorphism.

*Proof.* By abuse of notation, we identify  $C$  and  $C'$ ,  $\mathcal{X}^{\text{ac}}$  and  $\tilde{\mathcal{X}}^{\text{ac}}$ . Using the intersection formula, we have that

$$\text{DF}(\mathcal{X}'/C, -K_{\mathcal{X}'/C}) = -\frac{1}{2(n+1)(-K_{\mathcal{X}'_t})^n}(-K_{\mathcal{X}'/C})^{n+1}.$$

Similarly,

$$\text{DF}(\mathcal{X}^{\text{s}}/C, -K_{\mathcal{X}^{\text{s}}/C}) = -\frac{1}{2(n+1)(-K_{\mathcal{X}^{\text{s}}_t})^n}(-K_{\mathcal{X}^{\text{s}}/C})^{n+1}.$$

Let  $p: \hat{\mathcal{X}} \rightarrow \mathcal{X}'$  and  $q: \hat{\mathcal{X}} \rightarrow \mathcal{X}^{\text{s}}$  be a common log resolution, and write

$$(\pi' \circ p)^*(K_{\mathcal{X}^{\text{ac}}}) = p^*K_{\mathcal{X}'} = q^*K_{\mathcal{X}^{\text{s}}} + E.$$

Since  $\mathcal{X}' \dashrightarrow \mathcal{X}^{\text{s}}$  is a birational contraction, by negativity lemma (cf. [KM98, 3.39]), we conclude that  $E \geq 0$ . For  $0 \leq \lambda \leq 1$ , let

$$f(\lambda) = (-p^*K_{\mathcal{X}'/C} + \lambda E)^{n+1}.$$

Then for any  $0 \leq \lambda \leq 1$ ,

$$\begin{aligned} \frac{df(\lambda)}{d\lambda} &= (n+1)E \cdot (-p^*K_{\mathcal{X}'} + \lambda E)^n \\ &= (n+1)E \cdot (-(1-\lambda)p^*K_{\mathcal{X}'} - \lambda q^*K_{\mathcal{X}^{\text{s}}})^n \\ &\geq 0 \end{aligned}$$

since  $-(1-\lambda)p^*K_{\mathcal{X}'} - \lambda q^*K_{\mathcal{X}^{\text{s}}}$  is relatively nef over  $C$ . Thus

$$\text{DF}(\mathcal{X}'/C', -K_{\mathcal{X}'}) \geq \text{DF}(\mathcal{X}^{\text{s}}/C', -K_{\mathcal{X}^{\text{s}}}).$$

We analyze when the equality holds. If  $E = 0$ , then

$$\mathcal{X}^{\text{ac}} \cong \text{Proj } R(\mathcal{X}'/C, -K_{\mathcal{X}'/C}) \cong \text{Proj } R(\mathcal{X}^{\text{s}}/C, -K_{\mathcal{X}^{\text{s}}/C}) = \mathcal{X}^{\text{s}}.$$

So we may assume that the effective  $\mathbb{Q}$ -divisor  $E$  is not equal to 0.

Next we assume that  $\mathcal{X}^{\text{ac}}$  is isomorphic to  $\mathcal{X}^{\text{s}}$  in codimension 1. Thus for any divisor  $D$  on  $\mathcal{X}^{\text{ac}}$ ,

$$R(\mathcal{X}^{\text{ac}}/C, D) \cong R(\mathcal{X}^{\text{s}}/C, D_{\mathcal{X}^{\text{s}}}),$$

where  $D_{\mathcal{X}^{\text{s}}}$  is the push forward of  $D$  to  $\mathcal{X}^{\text{s}}$ . In particular, if we let  $D = -K_{\mathcal{X}^{\text{ac}}}$ , we again have

$$\mathcal{X}^{\text{ac}} \cong \text{Proj } R(\mathcal{X}^{\text{ac}}/C, -K_{\mathcal{X}^{\text{ac}}/C}) \cong \text{Proj } R(\mathcal{X}^{\text{s}}/C, -K_{\mathcal{X}^{\text{s}}/C}) = \mathcal{X}^{\text{s}}.$$

Thus we can assume that  $E > 0$  and  $\mathcal{X}^{\text{ac}}$  is not isomorphic to  $\mathcal{X}^{\text{s}}$  in codimension 1. Then we claim that

$$(5) \quad f(0) < f(\lambda)$$

for any  $1 > \lambda > 0$ .

In fact, since  $\mathcal{X}_0^s$  is irreducible, from the above discussion we may assume that there exists a component  $E_1 \subset \mathcal{X}_0^{\text{ac}}$  such that the birational transform  $\hat{E}_1$  of  $E_1$  on  $\hat{\mathcal{X}}$  is contracted under  $\hat{\mathcal{X}} \rightarrow \mathcal{X}^s$ . As  $-K_{\mathcal{X}^{\text{ac}}}$  is ample on  $E_1$ ,  $-(\pi' \circ p)^* K_{\mathcal{X}^{\text{ac}}}$  is nontrivial on the generic fiber of  $\hat{E}_1 \rightarrow \text{center}_{\mathcal{X}^s}(E_1)$ . This implies  $\hat{E}_1 \subset \text{Supp}(E)$  (cf. [KM98, 3.39]). Denote the coefficient of  $\hat{E}_1$  in  $E$  to be  $a > 0$ . Then

$$\begin{aligned} \left. \frac{df(\lambda)}{d\lambda} \right|_{\lambda=0} &= (n+1)E \cdot (-p^* K_{\mathcal{X}'})^n \\ &\geq (n+1)a\hat{E}_1 \cdot (-p^* K_{\mathcal{X}'})^n \\ &= (n+1)aE_1 \cdot (-K_{\mathcal{X}^{\text{ac}}})^n \\ &> 0. \end{aligned}$$

Since  $f(\lambda)$  is nondecreasing on  $\lambda \in [0, 1]$  and its derivative at 0 is positive, we easily see  $f(\lambda) > f(0)$  for any  $\lambda \in (0, 1]$ .  $\square$

## 6. Proof of Theorem 1

In this section, we finish proving Theorem 1 by combining the three steps proved in Theorem 2, Theorem 3 and Theorem 4.

*Proof of Theorem 1.* Let  $(\mathcal{X}, \mathcal{L})$  be any polarized generic  $\mathbb{Q}$ -Fano family. Note that, in particular, we assume  $\mathcal{X}$  is normal.

Then it follows from Theorem 2 that, after a base change  $\phi : C' \rightarrow C$ , we get a polarized generic  $\mathbb{Q}$ -Fano family  $(\mathcal{X}^{\text{lc}}, \mathcal{L}^{\text{lc}})$  satisfying the properties stated in Theorem 2. Letting  $l > 0$  be a sufficiently large integer, we can run a sequence of  $K_{\mathcal{X}^{\text{lc}}}$ -MMP over  $C$  with scaling of  $\mathcal{H}^{\text{lc}} = \mathcal{L}^{\text{lc}} - (l+1)^{-1}(K_{\mathcal{X}^{\text{lc}}} + \mathcal{L}^{\text{lc}})$  as in Section 4. We obtain a model  $\mathcal{X}^k \rightarrow C'$  where  $-K_{\mathcal{X}^k}$  is relatively big and semi-ample. Therefore, it admits an anti-canonical model  $\mathcal{X}^{\text{ac}}$  that satisfies the condition that  $(\mathcal{X}^{\text{ac}}, \mathcal{X}_t^{\text{ac}})$  is log canonical for every  $t \in C'$ . Finally, after a base change  $C'' \rightarrow C'$ , we construct a  $\mathbb{Q}$ -Fano family  $\mathcal{X}^s \rightarrow C''$ . After base change to  $C''$ , all our models after base change are isomorphic over  $C^* \times_C C''$ .

For the DF invariants,

$$\begin{aligned} (6) \quad \deg(C'/C) \cdot \text{DF}(\mathcal{X}/C, \mathcal{L}) &\geq \text{DF}(\mathcal{X}^{\text{lc}}/C', \mathcal{L}^{\text{lc}}) && \text{(by Theorem 2)} \\ (7) \quad &\geq \text{DF}(\mathcal{X}^{\text{ac}}/C', -K_{\mathcal{X}^{\text{ac}}}) && \text{(by Theorem 3)} \\ (8) \quad &\geq \frac{1}{\deg(C''/C')} \text{DF}(\mathcal{X}^s/C'', -K_{\mathcal{X}^s}) && \text{(by Theorem 4)}. \end{aligned}$$

By Theorem 4, the equality in (8) holds if and only if  $\tilde{\mathcal{X}}^{\text{ac}} = \mathcal{X}^s$ . Assume that  $t'' \in C''$  is mapped to  $t' \in C'$ ; then  $(\mathcal{X}^{\text{ac}}, \mathcal{X}_{t'}^{\text{ac}})$  is plt if and only if  $(\tilde{\mathcal{X}}^{\text{ac}}, \tilde{\mathcal{X}}_{t''}^{\text{ac}})$  (see [KM98, 5.20]), which then implies that  $\mathcal{X}^{\text{ac}}$  is a  $\mathbb{Q}$ -Fano family over  $C'$ .

By Theorem 3, the equality in (7) holds if and only if  $\mathcal{X}^{\text{lc}} = \mathcal{X}^{\text{ac}}$  and  $\mathcal{L}^{\text{lc}} = \mathcal{L}^{\text{ac}}$ .

Finally by Theorem 2, the equality in (6) holds if and only if  $(\mathcal{X}, \mathcal{X}_t)$  is log canonical for any  $t \in C$  and  $\mathcal{X}^{\text{lc}} \cong \mathcal{X} \times_C C'$ . As  $(\mathcal{X}^{\text{lc}}, \mathcal{X}_t^{\text{lc}}) \cong (\mathcal{X}^{\text{s}}, \mathcal{X}_t^{\text{s}})$  is a plt for any  $t \in C'$  and  $\mathcal{L}^{\text{lc}} \sim_{\mathbb{Q}, C'} -K_{\mathcal{X}^{\text{lc}}}$ , this implies that  $(\mathcal{X}, \mathcal{X}_t)$  is plt for any  $t \in C$  (cf. [KM98, 5.20]) and  $\mathcal{L} \sim_{\mathbb{Q}, C} -K_{\mathcal{X}}$ ; i.e.,  $\mathcal{X}$  is a  $\mathbb{Q}$ -Fano family over  $C$ .  $\square$

## Part 2. Application to KE Problem

In this part, we consider the application of Theorem 1 to the study of K-stability of Fano varieties and existence of Kähler-Einstein metric.

### 7. Introduction: K-stability from Kähler-Einstein problem

A fundamental problem in Kähler geometry is to determine whether there exists a Kähler-Einstein metric on a Fano manifold  $X$ , i.e., to find a Kähler metric  $\omega_{\text{KE}}$  in the Kähler class  $c_1(X)$  satisfying the equation

$$\text{Ric}(\omega_{\text{KE}}) = \omega_{\text{KE}}.$$

This is a variational problem. Futaki [Fut83] found an important invariant as the obstruction to this problem. Then Mabuchi [Mab86] defined the K-energy functional by integrating this invariant. This is a Kempf-Ness type function on the infinite dimensional space of Kähler metrics in  $c_1(X)$ . The minimizer of the K-energy is the Kähler-Einstein metric. Tian [Tia97] proved that, under some restriction on the automorphism group, there is a Kähler-Einstein metric if and only if the K-energy is proper on the space of all Kähler metrics in  $c_1(X)$ . So the problem is how to test the properness of the K-energy.

Tian also developed a program to reduce this infinite dimensional problem to finite dimensional problems. More precisely, he proved in [Tia90] that the space of Kähler metrics in a fixed Kähler class can be approximated by a sequence of spaces consisting of Bergman metrics. The latter spaces are finite dimensional symmetric spaces. Tian ([Tia97]) then introduced the K-stability condition using the generalized Futaki invariant ([DT92]) for testing the properness of K-energy on these finitely dimensional spaces. Later Donaldson [Don02] reformulated it by defining the Futaki invariants algebraically (see (9)), which is now called *the Donaldson-Futaki invariant*. The following folklore conjecture is the guiding question in this area.

**CONJECTURE 1** (Yau-Tian-Donaldson conjecture). *Let  $X$  be a Fano manifold. Then there is a Kähler-Einstein metric in  $-c_1(X)$  if and only if  $(X, -K_X)$  is K-polystable.*<sup>1</sup>

<sup>1</sup>In the recent remarkable works, Tian and Chen-Donaldson-Sun announced proofs of this conjecture [Tiaa], [CDSa], [CDSb], [CDSc]. Their results also imply Corollary 1 below.

See [Tia12] for more detailed discussion for the Kähler-Einstein problem.

In the following we will recall the definition of K-stability. First we need to define the notion of *test configuration*.

*Definition 3.* (1) Let  $X$  be an  $n$ -dimensional  $\mathbb{Q}$ -Fano variety. Assume that  $-rK_X$  is Cartier for some fixed  $r \in \mathbb{N}$ . A test configuration of  $(X, -rK_X)$  consists of

- a variety  $\mathcal{X}^{\text{tc}}$  with a  $\mathbb{G}_m$ -action;
- a  $\mathbb{G}_m$ -equivariant ample line bundle  $\mathcal{L}^{\text{tc}} \rightarrow \mathcal{X}^{\text{tc}}$ ;
- a flat  $\mathbb{G}_m$ -equivariant map  $\pi : (\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) \rightarrow \mathbb{A}^1$ , where  $\mathbb{G}_m$  acts on  $\mathbb{A}^1$  by multiplication in the standard way  $(t, a) \rightarrow ta$

such that for any  $t \neq 0$ ,  $(\mathcal{X}_t^{\text{tc}}, \mathcal{L}_t^{\text{tc}})$  is isomorphic to  $(X, -rK_X)$ , where  $\mathcal{X}_t^{\text{tc}} = \pi^{-1}(t)$  and  $\mathcal{L}_t^{\text{tc}} = \mathcal{L}^{\text{tc}}|_{\mathcal{X}_t^{\text{tc}}}$ .

(2) Fix  $r \in \mathbb{Q}_{>0}$ . We call  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  a  $\mathbb{Q}$ -test configuration of  $(X, -rK_X)$  if  $\mathcal{L}^{\text{tc}}$  is a  $\mathbb{Q}$ -Cartier divisor class on  $\mathcal{X}^{\text{tc}}$  such that for some integer  $m \geq 1$ ,  $(\mathcal{X}^{\text{tc}}, m\mathcal{L}^{\text{tc}})$  yields a test configuration of  $(X, -mrK_X)$ .

Obviously we can rescale the polarization of any test configuration to obtain a  $\mathbb{Q}$ -test configuration of  $(X, -K_X)$ .

Similarly to the notion of  $\mathbb{Q}$ -Fano family, we have the following definition.

*Definition 4.* A normal  $\mathbb{Q}$ -test configuration  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  of  $(X, -K_X)$  is called a *special  $\mathbb{Q}$ -test configuration* if  $\mathcal{L}^{\text{tc}} \sim_{\mathbb{Q}} -K_{\mathcal{X}^{\text{tc}}}$  and  $\mathcal{X}_0^{\text{tc}}$  is a  $\mathbb{Q}$ -Fano variety. Whenever there is no ambiguity, we will also abbreviate it as a special test configuration.

For any  $\mathbb{Q}$ -test configuration, we can define *the Donaldson-Futaki invariant*. First by the Riemann-Roch theorem, for sufficiently divisible  $k \in \mathbb{N}$ , we have

$$d_k = \dim H^0(X, \mathcal{O}_X(-kK_X)) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$$

for some rational numbers  $a_0$  and  $a_1$ . Let  $(\mathcal{X}_0^{\text{tc}}, \mathcal{L}_0^{\text{tc}})$  be the restriction of  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  over  $\{0\}$ . Since  $\mathbb{G}_m$  acts on  $(\mathcal{X}_0^{\text{tc}}, \mathcal{L}_0^{\text{tc} \otimes k})$ ,  $\mathbb{G}_m$  also acts on the space of holomorphic sections  $H^0(\mathcal{X}_0^{\text{tc}}, k\mathcal{L}_0^{\text{tc}})$ . We denote the total weight of this action by  $w_k$ . By the equivariant Riemann-Roch Theorem,

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

So we can expand

$$\frac{w_k}{kd_k} = F_0 + F_1 k^{-1} + O(k^{-2}).$$

*Definition 5* ([Don02]). The (normalized) Donaldson-Futaki invariant (DF-invariant) of the  $\mathbb{Q}$ -test configuration  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  is defined to be

$$(9) \quad \text{DF}(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) = -\frac{F_1}{a_0} = \frac{a_1 b_0 - a_0 b_1}{a_0^2}$$

With the normalization in (9), we easily see for any  $a \in \mathbb{Q}_{>0}$ ,  $\text{DF}(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) = \text{DF}(\mathcal{X}^{\text{tc}}, a\mathcal{L}^{\text{tc}})$ .

*Remark 1.* We have the following remarks about Donaldson-Futaki invariant:

- (1) From the differential geometry side, Ding and Tian [DT92] defined the generalized Futaki invariants by extending the original differential geometric formula of Futaki [Fut83] from smooth manifolds to normal varieties. On a normal  $\mathbb{Q}$ -Fano variety, the differential geometric definition coincides with the above algebraic definition. This was proved by Donaldson [Don02] in the smooth case. The calculation via equivariant forms in [Don02] is also valid in the normal case, because the codimension of singularities on a normal variety is at least two.
- (2) In [PT] and [PT09], Paul and Tian proved that the Donaldson-Futaki invariant is the same as the total  $\mathbb{G}_m$ -weight of the CM line bundle, which was introduced to give a GIT formulation of K-stability. See [FS90], [Tia97], [PT], [PT09] for details.

As we will show in Section 8.1, by adding a ‘trivial fiber’ over the point  $\infty \in \mathbb{P}^1$ , we can compactify a  $\mathbb{Q}$ -test configuration  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) \rightarrow \mathbb{A}^1$  to obtain a polarized generic  $\mathbb{Q}$ -Fano family  $(\bar{\mathcal{X}}^{\text{tc}}/\mathbb{P}^1, \bar{\mathcal{L}}^{\text{tc}})$ . By comparing the DF invariant and DF invariant, we have the equality

$$\text{DF}(\bar{\mathcal{X}}^{\text{tc}}/\mathbb{P}^1, \bar{\mathcal{L}}^{\text{tc}}) = \text{DF}(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}),$$

which explains the origin of our terminology.

If we apply our Theorem 1 to this case, it specializes to the following result.

**THEOREM 7.** *Let  $X$  be a  $\mathbb{Q}$ -Fano variety and  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  a test configuration of  $(X, -K_X)$ . We can construct a special test configuration  $(\mathcal{X}^{\text{st}}, -K_{\mathcal{X}^{\text{st}}})$  and a positive integer  $m$  such that*

$$m\text{DF}(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) \geq \text{DF}(\mathcal{X}^{\text{st}}, -K_{\mathcal{X}^{\text{st}}}).$$

*Furthermore, if we assume that  $\mathcal{X}^{\text{tc}}$  is normal, then the equality holds for our construction only when  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  itself is a special test configuration.*

This corollary will be applied to study K-stability, which we define as follows.

*Definition 6.* Let  $X$  be a  $\mathbb{Q}$ -Fano variety.

- (1)  $X$  is called K-semistable if for any  $\mathbb{Q}$ -test configuration  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  of  $(X, -K_X)$ , we have  $\text{DF}(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) \geq 0$ .
- (2)  $X$  is called K-stable (resp. K-polystable) if for any normal  $\mathbb{Q}$ -test configuration  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  of  $(X, -K_X)$ , we have  $\text{DF}(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) \geq 0$ , and the equality holds only if  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  is trivial (resp. only if  $\mathcal{X}^{\text{tc}} \cong X \times \mathbb{A}^1$ ).

*Remark 2.* We have the following remarks for the definitions of K-stability and K-polystability.

- (1) Though the notions of K-stability can be stated for a general singular variety  $X$  with  $-K_X$  being  $\mathbb{Q}$ -Cartier and ample, in [Oda13b], Odaka shows that for such a variety, if it is K-semistable, it can only have klt singularities.
- (2) In the definitions of K-polystability and K-stability, for the triviality of the test configuration with Donaldson-Futaki invariant 0, we require the test configuration to be normal. This is slightly different with the original definition. However, we believe this should be the right one. For more details, see Section 8.2. It is a consequence of [RT07, 5.2] that we only need to consider normal test configurations for K-semistability as well.

All these notions of test configurations, Donaldson-Futaki invariants and K-(semi,poly)-stability can be defined for a general polarized projective variety  $(X, L)$ . A more general version of Conjecture 1 predicts the equivalence between the K-polystability of a polarized manifold  $(X, L)$  and the existence of a constant scalar curvature Kähler metric in  $c_1(L)$ . Nevertheless, in this paper, except in Section 8.2 we mainly consider the notion of K-stability for the Kähler-Einstein problem on Fano varieties.

In [Tia97], where Tian gave the original definition of the K-stability in the analytic setting, he only considered test configurations with normal central fibers. Later he conjectured that (see [Tiab]) for Fano manifolds, even with Donaldson's definition, one still only needs to consider those test configurations with normal central fibers. This is motivated by compactness results for Kähler-Einstein manifolds (See [CCT02]).

As an immediate consequence of Corollary 7, we verify Tian's conjecture. In fact, it suffices to consider an even smaller class of test configurations, namely, the test configurations whose central fibers are  $\mathbb{Q}$ -Fano.

**COROLLARY 1 (Tian's conjecture).** *Assume that  $X$  is a  $\mathbb{Q}$ -Fano variety. If  $X$  is destabilized by a test configuration, then  $X$  is indeed destabilized by a special test configuration. More precisely, the following two statements are true:*

- (1) (K-semistability). *If  $X$  is not K-semi-stable, then there exists a special  $\mathbb{Q}$ -test configuration  $(\mathcal{X}^{\text{st}}, -K_{\mathcal{X}^{\text{st}}})$  with a negative Futaki invariant  $\text{DF}(\mathcal{X}^{\text{st}}, -K_{\mathcal{X}^{\text{st}}}) < 0$ .*

- (2) (K-polystability). *Let  $X$  be a K-semistable variety. If  $X$  is not K-polystable, then there exists a special  $\mathbb{Q}$ -test configuration  $(\mathcal{X}^{\text{st}}, -K_{\mathcal{X}^{\text{st}}})$  with Donaldson-Futaki invariant 0 such that  $\mathcal{X}^{\text{st}}$  is not isomorphic to  $X \times \mathbb{A}^1$ .*

### 8. Donaldson-Futaki invariant and K-stability

In this section, we will concentrate on the study of Donaldson-Futaki invariants of a test configuration, which is algebraically defined by Donaldson [Don02]. In the first subsection, we recall the fact that for a given test configuration  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) \rightarrow \mathbb{A}^1$ , its Donaldson-Futaki invariant coincides with the Donaldson-Futaki invariant of the natural compactification  $(\bar{\mathcal{X}}^{\text{tc}}, \bar{\mathcal{L}}^{\text{tc}}) \rightarrow \mathbb{P}^1$ . This characterization of the Donaldson-Futaki invariants first appears in Wang’s work [Wan12]. A different proof was also given in [Oda13a]. In the second subsection, we correct a small inaccuracy in the original definition of K-polystability in literatures.

8.1. *Intersection formula for the Donaldson-Futaki invariant.* Given any test configuration  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$ , we first compactify it by gluing  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  with  $(X \times (\mathbb{P}^1 \setminus \{0\}), p_1^*L)$ . It is known that the Donaldson-Futaki invariant is equal to the DF invariant on this compactified space as defined in Definition 2 (see [Wan12, Oda13a]). We will present a proof for reader’s convenience.

*Example 3.*  $\mathbb{G}_m$  acts on  $(X, L^{-1}) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))$  by

$$t \circ ([Z_0, Z_1], \lambda(Z_0, Z_1)) = ([Z_0, tZ_1], \lambda(Z_0, tZ_1)).$$

In particular, the  $\mathbb{G}_m$ -weights on

$$\mathcal{O}_{\mathbb{P}^1}(-1)|_0, \mathcal{O}_{\mathbb{P}^1}(1)|_0, \mathcal{O}_{\mathbb{P}^1}(-1)|_\infty \text{ and } \mathcal{O}_{\mathbb{P}^1}(1)|_\infty$$

are 0,0,1 and -1. Let  $\tau_0 = Z_1, \tau_\infty = Z_0$  be the holomorphic sections of  $\mathcal{O}_{\mathbb{P}^1}(1)$ . Then the  $\mathbb{G}_m$ -weights of  $\tau_0$  and  $\tau_\infty$  are -1 and 0.

Take  $\bar{\mathcal{X}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$  and  $\bar{\mathcal{L}} = \mathcal{O}_{\bar{\mathcal{X}}}(1) = \mathcal{O}_{D_\infty}$ , where  $D_\infty$  is the divisor at infinity. We see that  $(\bar{\mathcal{X}}^{\text{tc}} := \bar{\mathcal{X}} \setminus \mathbb{P}_\infty^1, \bar{\mathcal{L}}^{\text{tc}} := \bar{\mathcal{L}}|_{\bar{\mathcal{X}}^{\text{tc}}})$  yields a test configuration of  $(X, L)$ . Then  $H^0(\mathbb{P}^1, L^{\otimes k})$  is of dimension  $d_k = k + 1$  and by the calculation in the first paragraph, the total  $\mathbb{G}_m$ -weight of  $H^0(\mathbb{P}^1, L^{\otimes k})$  is  $w_k = -\frac{1}{2}(k^2 + k)$ . We know  $D_\infty^2 = -1$  and  $K_{\bar{\mathcal{X}}}^{-1} \cdot D_\infty = 1$ . So

$$w_k = \frac{D_\infty^2}{2}k^2 + \left( \frac{K_{\bar{\mathcal{X}}}^{-1} \cdot D_\infty}{2} - 1 \right) k,$$

$$\text{DF}(\bar{\mathcal{X}}^{\text{tc}}, \bar{\mathcal{L}}^{\text{tc}}) = \frac{D_\infty^2}{2} - \left( \frac{K_{\bar{\mathcal{X}}}^{-1} \cdot D_\infty}{2} - 1 \right) (= 0).$$

This example illustrates general cases (see (11), (12)). In the following we will use Donaldson’s argument (see the proof of Proposition 4.2.1 in [Don02]) to get the general intersection formula for DF invariant.

First note that, after identifying the fiber  $\mathcal{X}_1^{\text{tc}}$  over  $\{1\}$  and  $X$ , we have an equivariant isomorphism:

$$(\mathcal{X}^{\text{tc}} \setminus \mathcal{X}_0^{\text{tc}}, \mathcal{L}^{\text{tc}}) \cong (X \times (\mathbb{A}^1 \setminus \{0\}), p_1^* L)$$

by  $(p, a, s) \rightarrow (a^{-1} \circ p, a, a^{-1} \circ s)$ . Therefore,  $\mathbb{G}_m$  acts on the right-hand side by

$$t \circ (\{p\} \times \{a\}, s) = (\{p\} \times \{ta\}, s)$$

for any  $p \in X$ ,  $a \in \mathbb{A}^1$  and  $s \in \mathcal{L}_p^{\text{tc}}$ . The gluing map is given by

$$\begin{array}{ccc} (\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) & & (X \times \mathbb{P}^1 \setminus \{0\}, p_1^* L) \\ \cup & & \cup \\ (\mathcal{X}^{\text{tc}} \setminus \mathcal{X}_0^{\text{tc}}, \mathcal{L}^{\text{tc}}) & \longrightarrow & (X \times (\mathbb{A}^1 \setminus \{0\}), p_1^* L) \\ (p, a, s) & \longmapsto & (\{a^{-1} \circ p\} \times \{a\}, a^{-1} \circ s), \end{array}$$

where  $\mathbb{G}_m$  only acts by multiplication on the factor  $\mathbb{P}^1 \setminus \{0\}$  of  $(X \times \mathbb{P}^1 \setminus \{0\}, p_1^* L)$ .

*Definition 7.* Using the above gluing map, from a test configuration  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  of  $(X, -rK_X)$ , we get a generic  $\mathbb{Q}$ -Fano family  $\bar{\pi} : (\bar{\mathcal{X}}^{\text{tc}}, \frac{1}{r}\bar{\mathcal{L}}^{\text{tc}}) \rightarrow \mathbb{P}^1$ , which we call  $\infty$ -trivial compactification of the test configuration.

In what follows, we will denote  $(\bar{\mathcal{X}}^{\text{tc}}, \bar{\mathcal{L}}^{\text{tc}})$  by  $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$  for simplicity. Note that there exists an integer  $N$  such that  $\bar{\mathcal{M}} = \bar{\mathcal{L}} \otimes \bar{\pi}^*(\mathcal{O}_{\mathbb{P}^1}(N \cdot \{\infty\}))$  is ample on  $\bar{\mathcal{X}}$  (cf. [KM98, 1.45]).

We need the following weak form of the Riemann-Roch formula whose proof is well known.

LEMMA 4. *Let  $X$  be an  $n$ -dimensional normal projective variety and  $L$  an ample divisor on  $X$  then*

$$\dim H^0(X, L^{\otimes k}) = \frac{L^n}{n!} k^n + \frac{1}{2} \frac{(-K_X) \cdot L^{n-1}}{(n-1)!} k^{n-1} + O(k^{n-2}).$$

We define

$$d_k = \dim H^0(X, L^{\otimes k}) =: a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$$

PROPOSITION 6. *Let  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}})$  be a test configuration of  $(X, -rK_X)$ . Assume that  $\mathcal{X}^{\text{tc}}$  is normal. Then*

$$(10) \quad \text{DF}(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) = \text{DF}(\bar{\mathcal{X}}/\mathbb{P}^1, \bar{\mathcal{L}}).$$

*Proof.* For  $k \gg 0$ , by the Serre Vanishing Theorem, we have two exact sequences:

$$\begin{array}{ccccccc}
& & A & & B & & C \\
& & \parallel & & \parallel & & \parallel \\
0 \longrightarrow & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}(-\mathcal{X}_0^{\text{tc}})) & \xrightarrow{\otimes \sigma_0} & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}) & \longrightarrow & H^0(\bar{\mathcal{X}}_0, \bar{\mathcal{M}}^{\otimes k}|_{\mathcal{X}_0^{\text{tc}}}) & \longrightarrow 0 \\
& & \parallel & & \parallel & & \parallel \\
0 \longrightarrow & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}(-\bar{\mathcal{X}}_\infty)) & \xrightarrow{\otimes \sigma_\infty} & H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}) & \longrightarrow & H^0(\bar{\mathcal{X}}_\infty, \bar{\mathcal{M}}^{\otimes k}|_{\bar{\mathcal{X}}_\infty}) & \longrightarrow 0, \\
& & \parallel & & \parallel & & \parallel \\
& & A & & B & & D
\end{array}$$

where  $\sigma_0, \sigma_\infty$  are sections of  $\bar{\pi}^* \mathcal{O}_{\mathbb{P}^1}(1)$  that are the pull back of the divisors  $\{0\}, \{\infty\}$  on  $\mathbb{P}^1$ .

We can assume that the  $\mathbb{G}_m$ -weights of  $\sigma_0$  and  $\sigma_\infty$  are  $-1$  and  $0$ . Note the first terms in the two exact sequences are the same as  $A := H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k} \otimes \bar{\pi}^* \mathcal{O}_{\mathbb{P}^1}(-1))$ . We have the equation:  $w_B = w_A - d_A + w_C = w_A + w_D$ , where we denote  $d_A$  and  $w_A$  to be the dimension and the  $\mathbb{G}_m$ -weight of the vector space  $A$  and similarly for  $d_B, w_C$  etc. Since the  $\mathbb{G}_m$ -weight of  $\mathcal{O}_{\mathbb{P}^1}(1)|_\infty$  is  $-1$  and  $\mathbb{G}_m$  acts on  $\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty}$  trivially, we have  $w_D = -kN \dim H^0(\bar{\mathcal{X}}_\infty, \bar{\mathcal{L}}^{\otimes k}|_{\bar{\mathcal{X}}_\infty})$ . So we get

$$w_C = d_A + w_D = d_B - d_C - kNd_D = d_B - (kN + 1)d_C.$$

Since  $\mathbb{G}_m$  acts trivially on  $\mathcal{O}_{\mathbb{P}^1}(1)|_0$ , we get the  $\mathbb{G}_m$ -weight on  $H^0(\mathcal{X}_0^{\text{tc}}, \bar{\mathcal{M}}^{\otimes k}|_{\mathcal{X}_0^{\text{tc}}}) = H^0(\mathcal{X}_0^{\text{tc}}, \mathcal{L}^{\text{tc} \otimes k}|_{\mathcal{X}_0^{\text{tc}}})$ :

$$w_k = \dim H^0(\bar{\mathcal{X}}, \bar{\mathcal{M}}^{\otimes k}) - (kN + 1) \dim H^0(\mathcal{X}_0^{\text{tc}}, \mathcal{L}^{\text{tc} \otimes k}|_{\mathcal{X}_0^{\text{tc}}}).$$

Expanding  $w_k$ , we get

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}),$$

with

$$(11) \quad b_0 = \frac{\bar{\mathcal{M}}^{n+1}}{(n+1)!} - Na_0 = \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)!}$$

and

$$(12) \quad b_1 = \frac{1}{2} \frac{(-K_{\bar{\mathcal{X}}}) \cdot \bar{\mathcal{M}}^n}{n!} - Na_1 - a_0 = \frac{1}{2} \frac{(-K_{\bar{\mathcal{X}}}) \cdot \bar{\mathcal{L}}^n}{n!} - a_0.$$

By substituting the coefficients into (9), we get

$$\begin{aligned} \frac{a_1 b_0 - a_0 b_1}{a_0^2} &= \frac{1}{(n+1)! a_0} \left( \frac{a_1}{a_0} \bar{\mathcal{L}}^{n+1} + \frac{n+1}{2} K_{\bar{\mathcal{X}}} \cdot \bar{\mathcal{L}}^n \right) + 1 \\ &= \frac{1}{(n+1)r^n (-K_X)^n} \left( \frac{n}{2r} \bar{\mathcal{L}}^{n+1} + \frac{n+1}{2} K_{\bar{\mathcal{X}}} \cdot \bar{\mathcal{L}}^n \right) + 1 \\ &= \frac{1}{2(n+1)(-K_X)^n} \left( n \left( \frac{1}{r} \bar{\mathcal{L}} \right)^{n+1} + (n+1) K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \left( \frac{1}{r} \bar{\mathcal{L}} \right)^n \right) \\ &= \text{DF}(\bar{\mathcal{X}}/\mathbb{P}^1, \bar{\mathcal{L}}). \quad \square \end{aligned}$$

*Remark 3.*

- (1) As the above proof shows, Donaldson’s formula of Futaki invariant in the toric case (Proposition 4.2.1 in [Don02]) is a special example of the intersection formula. This intersection formula is also related to the interpretation of Donaldson-Futaki invariant as the CM-weight in [PT].
- (2) When  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) \rightarrow \mathbb{A}^1$  is a test configuration, where we only assume  $\mathcal{L}^{\text{tc}}$  to be relative big and semi-ample  $\mathbb{Q}$ -line bundle, this definition of Donaldson-Futaki invariant using intersection numbers  $\text{DF}(\bar{\mathcal{X}}^{\text{tc}}/\mathbb{P}^1, \bar{\mathcal{L}}^{\text{tc}})$  still coincides with the definition via computing the  $\mathbb{G}_m$ -weights of cohomological groups as in [ADVLN12]. For more details, see [RT07] and [ADVLN12].

8.2. *Normal test configuration.* It follows from [RT07, 5.1] or [ADVLN12, 3.9] that if  $n : (\mathcal{X}', n^* \mathcal{L}) \rightarrow (\mathcal{X}, \mathcal{L})$  is a finite morphism between test configurations of a polarized projective variety  $(X, L)$  that is an isomorphism over  $\mathbb{C}^*$ , then

$$\text{DF}(\mathcal{X}', \mathcal{L}') \leq \text{DF}(\mathcal{X}, \mathcal{L}),$$

and the equality holds if and only if  $n$  is an isomorphism in codimension 1. So to make the definition of K-polystability reasonable, we need to identify the test configurations that are isomorphic in codimension 1. Alternatively, when  $X$  is normal (resp.  $S_2$ ), we could only consider test configurations  $\mathcal{X}$  that are normal (resp.  $S_2$ ) as in Definition 6.

The following easy example shows that pathological nontrivial test configurations, which are trivial in codimension 1, do occur.

*Example 4.* Let  $(X, L) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$ . Consider the test configuration

$$\mathcal{X} \subset \mathbb{P}^3 \times \mathbb{A}^1 = \mathbb{P}(x, y, z, w) \times \text{Spec } k[a]$$

given by

$$I = (a^2(x+w)w - z^2, ax(x+w) - yz, xz - ayw, y^2w - x^2(x+w))$$

(cf. [Har77, III.9.8.4]). The  $\mathbb{G}_m$  action on it is just sending

$$\mathcal{X} \times \mathbb{G}_m \rightarrow \mathcal{X} : (x, y, z, w; a) \times \{t\} \rightarrow (x, y, tz, w; at).$$

Then for  $a = 0$ , the special fiber has the ideal

$$I_0 = (z^2, yz, xz, y^2w - x^2(x + w)).$$

Geometrically,  $\mathcal{X}_t^{\text{tc}}$  is a cubic curve in  $\mathbb{P}^3$ . They degenerate to the special fiber  $\mathcal{X}_0^{\text{tc}}$  that is a plane nodal cubic curve in  $\mathbb{P}^2 = \mathbb{P}(x, y, w)$  with an embedded point at  $(0, 0, 0, 1)$ .

For  $k \gg 0$ , we have

$$h^0(\mathbb{P}^1, kL) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3k)) = 3k + 1 \quad \text{and} \quad H^0(\mathcal{X}_0, k\mathcal{L}_0) = V_1 \oplus V_2,$$

where

$$V_1 \cong H^0(\mathcal{X}_0^{\text{red}}, \mathcal{O}_{\mathbb{P}(x,y,w)}(k)|_{\mathcal{X}_0^{\text{red}}}),$$

and  $V_2$  is the one dimensional space spanned by  $z \cdot w^{k-1}$  (or  $z \cdot f(x, y, w)$  for any homogeneous polynomial of degree  $k-1$  such that  $f(0, 0, 1) \neq 0$ ). As the total weight of  $V_1$  is 0 and the total weight of  $V_2$  is 1, we conclude that  $b_0 = b_1 = 0$ .

In fact, it is easy to see in general such pathological test configuration exists for any polarized variety  $(X, L)$ .

*Remark 4.* (1) In [Sto09], J. Stoppa claimed a proof of the K-stability of varieties with Kähler-Einstein metric under the original definition, namely without assuming the normality of the test configuration. However, he made a mistake on the calculation of the Donaldson-Futaki invariant for the ‘degenerate case.’ More precisely, the formula (3.7) of the proof of Proposition 3.3 in [Sto09] is false because multiplying sections  $H^0(\mathcal{X}_0^{\text{red}}, \mathcal{L}^{k-1}|_{\mathcal{X}_0^{\text{red}}})$  by a nilpotent element is not always an injection in general.

There is also a similar overlooking in [PT09]. Corollary 2 there said that the properness of K-energy implies Donaldson-Futaki invariant is positive for any test configuration. The case missing in their proof in Section 3.3 is when  $\lim_{t \rightarrow 0} \text{Osc}(\phi_t) < \infty$  and the central fibre is generically reduced as the above example shows.

(2) As far as we can see, in most of the published literature, including [RT07], [Oda13b] and [Oda12], the same arguments of proving the results on K-stability for certain classes of varieties work, once we replace the definition there by our new definition. More precisely, for any nontrivial *normal* test configuration  $\mathcal{X}^{\text{tc}}$ , there is a semi-test configuration  $\mathcal{Y}^{\text{tc}}$  with equivariant morphisms  $p : \mathcal{Y}^{\text{tc}} \rightarrow \mathcal{X}^{\text{tc}}$  and  $q : \mathcal{Y}^{\text{tc}} \rightarrow X \times \mathbb{A}^1$  such that  $q$  is not the trivial morphism. Therefore,  $q$  gives an exceptional divisor  $E$  over  $X \times \mathbb{A}^1$ . Then their calculations can be carried out by using this exceptional divisor.

## 9. Proof of Theorem 7

To prove Theorem 7, now we only need to check that if we start with a  $\infty$ -trivial compactification of a test configuration as in Section 8.1, the families

in Theorems 2, 3 and 4 can be also constructed to be  $\infty$ -trivial compactifications of test configurations.

9.1. *Equivariant Semi-stable reduction.* The following result, whose proof is a simple combination of the equivariant resolution (see, e.g., [Kol07b, 3.9.1] and reference therein) and the semistable reduction (see [KKMSD73]), is well known. However, we cannot find it in the literature. Hence we include a short argument here. A similar statement for resolution appears in [ADVLN12].

LEMMA 5. *Let  $f : \mathcal{X} \rightarrow \mathbb{A}^1$  be a dominant morphism from a normal variety with a  $\mathbb{G}_m$ -action such that  $f$  is  $\mathbb{G}_m$ -equivariant. Then there exist a base change  $z^m : \mathbb{A}^1 \rightarrow \mathbb{A}^1$  and a semistable family  $\mathcal{Y}$  over  $\mathbb{A}^1$  with a morphism  $\pi : \mathcal{Y} \rightarrow \mathcal{X} \times_{\mathbb{A}^1, z^m} \mathbb{A}^1$  that is a log resolution of  $(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_0)$  where  $\tilde{\mathcal{X}}$  is the normalization of  $\mathcal{X} \times_{\mathbb{A}^1, z^m} \mathbb{A}^1$ .*

*Proof.* First, we perform the blow up of  $(\mathcal{X}, \mathcal{X}_0)$   $\mathbb{G}_m$ -equivariantly to get an equivariant log resolution  $\mathcal{Y}^*$ . This is always possible by the theorem of equivariant resolution of singularities. Then we can take a base change  $z^m : \mathbb{A}^1 \rightarrow \mathbb{A}^1$  such that the normalization  $\tilde{\mathcal{Y}}^*$  of  $\mathcal{Y}^* \times_{\mathbb{A}^1, z^m} \mathbb{A}^1$  has a reduced fiber over each point of  $\mathbb{A}^1$ . Then it follows from [KKMSD73] that possibly after a further base change, we can take a sequence of toroidal blow-ups of  $\tilde{\mathcal{Y}}^*$  to obtain a log resolution  $\mathcal{Y}$  of  $(\tilde{\mathcal{Y}}^*, \tilde{\mathcal{Y}}_0^*)$  such that  $\mathcal{Y}_0$  has reduced fibers.

As each component of  $\tilde{\mathcal{Y}}_0^*$  is  $\mathbb{G}_m$ -invariant, so are the irreducible components of their intersections. Since the centers of the toroidal blow-ups are  $\mathbb{G}_m$ -invariant,  $\mathbb{G}_m$  action on  $\tilde{\mathcal{Y}}^*$  can be successively lifted to  $\mathcal{Y}$ .  $\square$

9.2. *Proof of Tian’s conjecture.*

*Proof of Theorem 7.* It suffices to check that if we start with an  $\infty$ -trivial compactification

$$(\bar{\mathcal{X}}^{\text{tc}}, \bar{\mathcal{L}}^{\text{tc}}) \rightarrow \mathbb{P}^1$$

of a  $\mathbb{Q}$ -test configuration  $(\mathcal{X}^{\text{tc}}, \mathcal{L}^{\text{tc}}) \rightarrow \mathbb{A}^1$ , the models we construct in Theorems 2, 3 and 4 are all  $\infty$ -trivial compactifications of  $\mathbb{Q}$ -test configurations.

Since resolution of singularities and semistable reduction can be obtained  $\mathbb{G}_m$ -equivariantly (see Lemma 5), starting from an  $\infty$ -trivial compactification  $(\bar{\mathcal{X}}^{\text{tc}}, \bar{\mathcal{L}}^{\text{tc}}) \rightarrow \mathbb{P}^1$ , we can perform a base change  $d : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  such that  $\bar{\mathcal{X}}_d^{\text{tc}} := \bar{\mathcal{X}}^{\text{tc}} \times_{d, \mathbb{P}^1} \mathbb{P}^1 \rightarrow \mathbb{P}^1$  via the second projection admits a  $\mathbb{G}_m$ -equivariant semistable reduction  $\pi : \mathcal{Y} \rightarrow \bar{\mathcal{X}}_d^{\text{tc}}$ .

Then the log canonical modification

$$\mathcal{X}^{\text{lc}} = \text{Proj } R(\mathcal{Y}/\bar{\mathcal{X}}_d^{\text{tc}}, K_{\mathcal{Y}})$$

admits a  $\mathbb{G}_m$ -action such that  $\pi^{\text{lc}} : \mathcal{X}^{\text{lc}} \rightarrow \bar{\mathcal{X}}_d^{\text{tc}}$ , which is isomorphism over  $\mathbb{P}^1 \setminus \{0\}$ , is equivariant. Thus the polarization

$$\mathcal{L}^{\text{lc}} = (\pi^{\text{lc}})^* d^* \bar{\mathcal{L}}^{\text{tc}} + tK_{\mathcal{X}^{\text{lc}}}$$

for sufficiently small  $t > 0$  also clearly admits a compatible  $\mathbb{G}_m$ -action.

Now we run a relative  $K_{\mathcal{X}^{lc}}$ -MMP with scaling of  $\mathcal{L}^{lc}$  over  $\mathbb{P}^1$ . Each step is indeed automatically  $\mathbb{G}_m$ -equivariant. In fact, assuming this is true after the  $i$ -th step, since  $\mathbb{G}_m$  is connected, then  $\mathrm{NE}(X^i)^{\mathbb{G}_m} = \mathrm{NE}(X^i)$ . (See the proof of [And01, 1.5].) Hence the contraction is  $\mathbb{G}_m$ -equivariant. As the flip is a Proj of a  $\mathbb{G}_m$ -equivariant algebra, it also admits a  $\mathbb{G}_m$ -action. Therefore, at the end  $\mathcal{X}^{ac}$  is  $\mathbb{G}_m$ -equivariant. So  $(\mathcal{X}^{ac}, -K_{\mathcal{X}^{ac}})$  is a  $\infty$ -trivial compactification of the associated test configuration.

Similarly, the process involved in the proof of Theorem 4 can be proceeded  $\mathbb{G}_m$ -equivariantly so that the special  $\mathbb{Q}$ -Fano family we obtain at the end yields a special test configuration. We leave the details to the reader.  $\square$

*Proof of Corollary 1.* The semistability case follows from Theorem 1 immediately. Now we assume that  $(X, -rK_X)$  is  $K$ -semistable and all special test configurations  $(\mathcal{X}^{st}, \mathcal{L}^{st})$  with  $\mathrm{DF}(\mathcal{X}^{st}, \mathcal{L}^{st}) = 0$  satisfy that  $\mathcal{X}^{st} \cong X \times \mathbb{A}^1$ . Let  $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$  be an arbitrary normal test configuration whose DF invariant is 0. Applying Theorem 1, we obtain a special test configuration  $(\mathcal{X}^{st}, -K_{\mathcal{X}^{st}})$  of  $(X, -rK_X)$  and inequalities

$$0 \leq \mathrm{DF}(\mathcal{X}^{st}, -rK_{\mathcal{X}^{st}}) \leq m\mathrm{DF}(\mathcal{X}^{tc}, \mathcal{L}^{tc}) = 0.$$

Then since the equality holds, by the conclusion of Theorem 1 we know that  $(\mathcal{X}^{tc}, \mathcal{L}^{tc})$  is a special test configuration, which implies that  $\mathcal{X}^{tc} \cong X \times \mathbb{A}^1$ .  $\square$

We finish our article with the following remark.

*Remark 5.* We are inspired by Odaka's algebraic proof of  $K$ -stability of canonically polarized variety and Calabi-Yau variety [Oda12], which provides the counterpart of our theory for the case when  $K_X$  is ample or trivial.

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