MA 266 Lecture 1

Section 1.1 Mathematical Models; Direction Fields

Question: What is a differential equation?

A differential equation is equation containing derivatives.

Example 1. (Types of equations)

1. Find $x$ in $x^2 + 2x + 1 = 0.$

2. Find $f(t)$ in $f(t) \cos(t) = e^t - \sin(t).$

3. Find $y(t)$ in $y'' + 3y' = e^t.$

Question: Why do we study differential equations?

- Many principles or laws in physics are relations involving differential equations.
- In mathematical terms, relations are equations, and rates are derivatives. Equations containing derivatives are differential equations.
- A differential equation that describes certain physical process is often called a mathematical model.
Example 2. *(An example of mathematical model — A falling object)*

Consider an object with a mass $m$ falling near the sea level. Formulate a differential equation to model its motion.

- **Notations**
  - $m$ — mass of the object
  - $a$ — the acceleration
  - $F$ — net force exerted on the object

- **Physical Law:** Newton’s second law
  \[ F = ma = m \frac{du}{dt} \]

- **Forces that acted on the object**
  - Gravity: $mg$
  - Drag force: $-TV$, $T$ — drag coefficient
  \[ F = mg - TV \]
  \[ \Rightarrow m \frac{dv}{dt} = mg - TV \]

**Remark** The falling object model contains three constants: $m$, $g$, and $\gamma$.

- $m$ — mass
- $g$ — gravity constant
- $\gamma$ — drag coefficient
Direction Fields

We let $m = 10\, kg$ and $\gamma = 2\, kg/s$ in the falling object model, so it becomes

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}.$$ 

Basic idea of direction fields:

Direction field can be constructed by evaluating $f$ at each point of a rectangular grid. At each point, a short line segment is drawn whose slope is the value $f$ at that point.

How to construct Direction Fields?

If we let $v = 40$, then

$$\frac{dv}{dt} = 1.8$$

If we let $v = 50$, then

$$\frac{dv}{dt} = -0.2$$

Note that if $9.8 - \frac{v}{5} = 0$, then

$$v = 5 \times 9.8 = 49\, m/s,$$

which is the equilibrium solution or terminal velocity.
Remarks on Direction Fields

Direction fields are valuable tools in studying differential equations of the form

\[ \frac{dy}{dt} = f(t, y) \]  \hspace{1cm} \text{rate function}

Two things about direction fields

- we evaluate \( f(t, y) \) many times.
- we can use computer to draw direction field.

A MATLAB Implementation on Direction Fields

1. Download the MATLAB file \texttt{dfield8.m} from

\[ \text{http://math.rice.edu/~dfield} \]

2. Type \texttt{dfield8}, at MATLAB command window.

3. In the popup window, enter your differential equations, and the range of independent and dependent variables.

4. Hit \textbf{Proceed} to see the direction field of your differential equation.
Example 3. Draw a direction field of each of the following differential equations, then determine the behavior of the solution as $t \to \infty$.

(1) $y' = 3 - 2y$,  
(2) $y' = 3 + 2y$,  
(3) $y' = -y(5 - y)$. 

![Direction Fields]