MA 266 Lecture 20

Section 3.7  Mechanic and Electrical Vibrations (contd)

Review  For undamped free vibrations, the governing equation is

\[ m u'' + k u = 0 \]

Example 1. (Problem 6) A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s, and if there is no damping, determine the position \( u \) of the mass at any time \( t \). When does the mass first return to its equilibrium position?

\[ m = 100 g = 0.1 \text{ kg} \]

\[ k = \frac{L \cdot mg}{L} = \frac{0.1 \text{ kg} \cdot 9.8 \text{ m}^2/\text{s}^2}{0.05 \text{ m}} = 19.6 \text{ kg}^2/\text{s}^2 \]

\[ t = 0 \]

no external force, \( F(t) = 0 \)

\[ u_0 = 0.1 \text{ m}^2/\text{s} = 0.1 \text{ m/s} \]

\[ u(0) = 0. \]

\[ u'(0) = 10 \text{ cm/s} = 0.1 \text{ m/s} \]

C.E. \[ 0.1 v^2 + 19.6 = 0 \Rightarrow \left| v \right| = 14 \text{ cm/s} \]

\[ u(t) = A \cos(14t) + B \sin(14t) \]

I.C.: \[ (1) A = 0, \quad B = \frac{1}{14} \Rightarrow u(t) = \frac{1}{14} \sin(14t) \]

Period \[ T = \frac{2\pi}{14} = \frac{\pi}{7} \text{ s} \]

After half period \( \frac{\pi}{14} \), it returns to equilibrium position.
Damped Free Vibrations

If we include the effect of damping, the differential equation the motion becomes

\[ m u'' + \gamma u' + ku = 0. \]

C. E.: \[ mr^2 + \gamma r + k = 0. \]

\[ r_{1,2} = -\frac{\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}. \]

Depending the sign of \( \gamma^2 - 4km \) the solution \( u \) has one of the following forms

- if \( \gamma^2 - 4km > 0 \) two real distinct roots, \( r_1, \) and, \( r_2 \)
  \[ u = Ae^{r_1t} + Be^{r_2t}. \]

- if \( \gamma^2 - 4km = 0 \) two real repeated roots, \( r_1 = r_2 = -\frac{\gamma}{2m} \)
  \[ u = (A + Bt)e^{-\frac{\gamma t}{2m}}. \]

- if \( \gamma^2 - 4km < 0 \) two complex roots

Remark

\[ u(t) = e^{-\frac{\gamma t}{2m}}(A\cos(\omega t) + B\sin(\omega t)), \quad \omega = \sqrt{\frac{4km - \gamma^2}{2m}}. \]

The solution \( u \) tends to zero as \( t \to \infty \) because of damping.

The case where \( \gamma^2 - 4km < 0 \) is of most interest. The solution can be written as

\[ u(t) = Re^{-\frac{\gamma t}{2m}}e^{i(\omega t - \delta)}. \]

\[ |u(t)| \leq Re^{-\frac{\gamma t}{2m}}. \]

Displacement \( u(t) \) lies between two curves

\[ u = 1Re^{-\frac{\gamma t}{2m}}. \]
Example 2. (Problem 10) A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb·s/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in/s, find its position \( u \) at any time \( t \). Determine when the mass first returns to its equilibrium position.

\[
mass, \quad m = \frac{W}{g} = \frac{16 \text{ lb}}{32 \text{ ft/s}^2} = \frac{1}{2} \text{ lb·s}^2/\text{ft}
\]

damping, \( \tau = 2 \text{ lb·s}/\text{ft} \)

spring constant, \( k = \frac{mg}{L} = \frac{16 \text{ lb}}{3 \text{ in}} = \frac{16 \text{ lb}}{3 \cdot \frac{1}{12} \text{ ft}} = 64 \text{ lb/ft} \)

ODE: \( \frac{1}{2} u'' + 2u' + 64u = 0 \)

I.C.: \( u(0) = 0, \quad u'(0) = 3 \text{ in/s} = 3 \cdot \frac{1}{12} \text{ ft/s} = \frac{1}{4} \text{ ft/s} \)

C.E.: \( t^2 + 4t + 128 = 0 \)

\( t = -2 \pm 2\sqrt{31} \text{ ft/s} \)

The general solution is:

\( U = e^{-2t} (A \cos (2\sqrt{31} t) + B \sin (2\sqrt{31} t)) \)

I.C.: \( \Rightarrow A = 0, \quad B = \frac{1}{8\sqrt{31}} \)

\( U = \frac{1}{8\sqrt{31}} e^{-2t} \sin (2\sqrt{31} t) \)

\( U = 0 \Rightarrow \sin (2\sqrt{31} t) = 0 \Rightarrow 2\sqrt{31} t = n\pi, \quad n = 1, 2, ... \)

Cot \( n = 1, \quad t = \frac{n}{2\sqrt{31}} \)