MA 266 Lecture 21

Section 3.8  Forced Vibration

In this section, we consider the situation in which a periodic external force is applied to a spring-mass system.

Forced Vibration with Damping

Example 1. Suppose that the motion of a certain spring-mass system satisfies the differential equation:

\[ u'' + u' + 1.25u = 3\cos(t), \quad u(0) = 2, \quad u'(0) = 3. \]

Find the solution of this initial value problem and describe the behavior of the solution for large t.

C.E.: \[ r^2 + r + 1.25 = 0. \]

\[ r = -0.5 \pm i. \]

\[ u(t) = C_1 e^{-0.5t} \cos t + C_2 e^{-0.5t} \sin t. \]

Assume a particular solution of Eq. (1):

\[ U(t) = A \cos t + B \sin t. \]

\[ (0.25A + B) \cos t + (-A + 0.25B) \sin t = 3 \cos t, \]

\[ \begin{align*}
0.25A + B &= 3 \\
-A + 0.25B &= 0
\end{align*} \]

\[ \Rightarrow A = \frac{12}{17}, \quad B = \frac{48}{17}. \]

\[ U = C_1 e^{-0.5t} \cos t + C_2 e^{-0.5t} \sin t + \frac{12}{17} \cos t + \frac{48}{17} \sin t. \]

\[ U(0) = C_1 + \frac{12}{17} = 2 \]

\[ \Rightarrow C_1 = \frac{22}{17}. \]

\[ U'(0) = -\frac{1}{2} C_1 + C_2 + \frac{48}{17} = 3 \]

\[ \Rightarrow C_2 = \frac{14}{17}. \]

\[ u = \frac{22}{17} e^{-0.5t} \cos t + \frac{14}{17} e^{-0.5t} \sin t + \frac{12}{17} \cos t + \frac{48}{17} \sin t. \]
In general, the equation of motion of a spring-mass system subject to an external force $F(t)$ is

$$U = C_1 u_1(t) + C_2 u_2(t) + A \cos \omega t + B \sin \omega t.$$  

**Forced Vibration without Damping**

In this case, the motion is governed by the equation

$$mu'' + ku = F_0 \cos(\omega t).$$

- If $\omega \neq \omega_0$,  
  $$U = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$ 

- If $\omega = \omega_0$,  
  $$U = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{2m \omega_0} t \sin \omega t.$$ 

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Example 2. (Problem 10) A mass weighing 8 lb stretches a spring 6 in. The mass is acted on by an external force of $8 \sin(8t)$ lb. If the mass is pulled down 3 in and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.

$$m = \frac{8 \text{ lb}}{32 \text{ ft/s}^2} = \frac{1}{4} \text{ lb/ft}$$

$$k = \frac{8 \text{ lb}}{\frac{1}{2} \text{ ft}} = 16 \text{ lb/ft}$$

$$\frac{1}{4} u'' + 16u = 8 \sin(8t)$$

$$u(0) = \frac{1}{4} \text{ ft}, \quad u'(0) = 0$$

$$u = C_1 \cos 8t + C_2 \sin 8t - \frac{8t \cos 8t}{2.48}$$

$$= C_1 \cos 8t + C_2 \sin 8t - 2t \cos 8t$$

$$u(0) = \frac{1}{4} = C_1 + 0 + 0 \Rightarrow C_1 = \frac{1}{4}$$

$$u'(0) = 0 = 0 + 8C_2 + (-2) = 0 \Rightarrow C_2 = \frac{1}{4}$$

$$\boxed{u = \frac{1}{4} [\cos 8t + \sin 8t - 8t \cos 8t]}$$

$$u' = 0 \Rightarrow -8 \sin 8t + 8 \cos 8t - 8 \cos 8t + 64t \sin 8t = 0$$

$$\sin 8t (8t - 1) = 0$$

$$t = \frac{1}{8}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \quad u' = 0$$