6.6 The Convolution Integral

In this section, we introduce an important tool for Laplace transform, which is known as the convolution.

**Question:** What is the inverse Laplace of $H(s)$, if $H(s) = F(s)G(s)$?

It is not $f(t) \cdot g(t)$

If $F(s) = \mathcal{L}\{f(t)\}$, and $G(s) = \{g(t)\}$, then $H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}$, where

$$h(t) = \int_0^t f(t-\tau)g(\tau) \, d\tau = \int_0^t f(\tau)g(t-\tau) \, d\tau$$

**Remark**

- The function $h$ is not $f \cdot g$
- The function $h$ is called **convolution** of $f$ and $g$.

The convolution $f \ast g$ has many properties of the ordinary multiplication

- $f \ast g = g \ast f$
- $f \ast (g_1 \ast g_2) = f \ast g_1 \ast f \ast g_2$
- $f \ast (g \ast h) = (f \ast g) \ast h$
- $f \ast 0 = 0$

However, $f \ast 1 \neq f$. To see this, we let $f(t) = \cos(t)$.

$$f \ast 1(t) = \int_0^t \cos(\tau-\tau) \, d\tau = -\sin(\tau-\tau) \bigg|_{\tau=0}^{\tau=t} = \sin(t)$$
Example 1. Find the Laplace transform of

\[ f(t) = \int_0^t \sin(t - \tau) \cos(\tau) d\tau \]

\[ f(t) = (\sin t + \cos t)(t) \]

\[ \mathcal{L}\{f(t)\} = \frac{1}{s^2 + 1} \quad \mathcal{L}\{\sin t\} = \frac{s}{s^2 + 1} \]

\[ \mathcal{L}\{\cos t\} = \frac{s}{s^2 + 1} \]

\[ F(s) = \frac{s}{(s^2 + 1)^2} \]

Example 2. Find the inverse transform of

\[ H(s) = \frac{1}{s^2(s^2 + 1)} \]

\[ H(s) = \frac{1}{s^2} \cdot \frac{1}{s^2 + 1} \]

\[ \mathcal{L}\{G(t)\} = \frac{s}{s^2 + 1} \]

\[ f(t) = \sin t \quad g(t) = \sin t \]

\[ h(t) = (f \ast g)(t) \]

\[ = \int_0^t (t - \tau) \sin(\tau) d\tau \]

\[ = \int_0^t t \sin(\tau) d\tau - \int_0^t \sin(\tau) d\tau \]

\[ = -t \cos(t) \bigg|_0^t - \left( -t \cos(t) \bigg|_0^t + \int_0^t \cos(\tau) d\tau \right) \]

\[ = -t \cos(t) \bigg|_0^t - \left( -t \cos(t) + \sin(t) \right) \]

\[ = t - s\sin(t) \]
Example 3. Find the solution of the initial value problem

\[ y'' + 4y = g(t), \quad y(0) = 3, \quad y'(0) = -1 \]

\[ s^2 Y(s) - 3s + 1 + 4Y(s) = G(s) \]

\[ (s^2 + 4) Y(s) = 3s - 1 + G(s) \]

\[ Y(s) = \frac{3s - 1}{s^2 + 4} + \frac{G(s)}{s^2 + 4} \]

\[ Y(s) = \frac{3}{s^2 + 4} - \frac{1}{2} \cdot \frac{2}{s^2 + 4} + \frac{1}{2} \cdot \frac{2}{s^2 + 4} \cdot G(s) \]

\[ y(t) = 3 \cos(2t) - \frac{1}{2} \sin(2t) + \frac{1}{2} \int_{0}^{t} \sin(2(t - \tau) - c) \cdot g(\tau) d\tau \]