7.3 Linear Dependence, Eigenvalues, Eigenvectors

Linear Dependence

A set of \( k \) vectors \( x^{(1)}, \ldots, x^{(k)} \) is said to be linearly dependent if there exists a set of real or complex numbers \( c_1, c_2, \ldots, c_k \), at least one of which is non-zero, such that

\[
\sum_{i=1}^{k} c_i x^{(i)} = 0
\]

On the other hand, if the only set \( c_1, \ldots, c_k \) satisfying the equation is

\[
c_1 = c_2 = \cdots = c_k = 0
\]

then \( x^{(1)}, \ldots, x^{(k)} \) is said to be linearly independent.

For vector functions \( x^{(1)}(t), \ldots, x^{(k)}(t) \), they are said to be linearly dependent on \( \alpha < t < \beta \) if there exists a set of constants \( c_1, c_2, \ldots, c_k \) s.t.

\[
\sum_{i=1}^{k} c_i x^{(i)}(t) = 0 \\
\text{for all } t \in (\alpha, \beta)
\]

Example 1. Verify that the following vectors \( x^{(1)}(t) \) and \( x^{(2)}(t) \) are linearly independent on the interval \( 0 < t < 1 \)

\[
x^{(1)}(t) = \begin{pmatrix} e^t \\ t e^t \end{pmatrix}, \quad x^{(2)}(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}
\]

\[
c_1 x^{(1)}(t) + c_2 x^{(2)}(t) = c_1 \begin{pmatrix} e^t \\ t e^t \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ t \end{pmatrix}
\]

\[
= \begin{pmatrix} c_1 e^t + c_2 \\ c_1 t e^t + c_2 t \end{pmatrix}
\]

\[
to \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\Rightarrow c_1 e^t + c_2 = 0 \quad \text{and} \quad c_1 t e^t + c_2 t = 0
\]

\[
\Rightarrow c_1 = 1, c_2 = 0
\]
Eigenvalues and Eigenfunctions

The equation \( Ax = y \) can be viewed as a linear transform that maps a given vector \( x \) into a new vector \( y \). Vectors that are transformed into multiples of themselves are important in many applications. To find such vectors, we let \( y = \lambda x \), then

\[
\begin{align*}
A \vec{x} &= \lambda \vec{x} \\
\begin{pmatrix} A - \lambda I \end{pmatrix} \vec{x} &= 0
\end{align*}
\]

\( \vec{x} \neq 0 \) is a solution (trivial)

\[
\det (A - \lambda I) = 0
\]

- characteristic equation of \( A \)
- value \( \lambda \) : eigenvalue
- \( \vec{x} \) : eigenvector associated with \( \lambda \)

Example 2. Find the eigenvalues and eigenvectors of the matrix

\[
A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}
\]

C. E. \[
\det (A - \lambda I) = \begin{vmatrix} 3 - \lambda & -1 \\ 4 & -2 - \lambda \end{vmatrix}
\]

\[\det (A - \lambda I) = 0 \quad (3 - \lambda)(-2 - \lambda) + 4 = 0\]

\( \lambda^2 + \lambda - 2 = 0 \)

\( (\lambda - 1)(\lambda + 2) = 0 \)

\( \lambda_1 = 1, \lambda_2 = -2 \)

For \( \lambda = 1 \):

\[
\begin{pmatrix} 3 - 1 & -1 \\ 4 & -2 - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\( 2x_1 - x_2 = 0 \)

\( 4x_1 = x_2 \)

Let \( x_2 = 4 \)

\( x_1 = 1 \)

\[ \vec{x}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \]

For \( \lambda = -2 \):

\[
\begin{pmatrix} 3 + 2 & -1 \\ 4 & -2 + 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\( 5x_1 - x_2 = 0 \)

\( x_1 = \frac{x_2}{5} \)

Choose \( x_2 = 1 \)

\( x_1 = 1 \)

\[ \vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
7.4 Theory of System of First Order Linear Equations

The general form of a system of \( n \) first order linear equations is

\[
\begin{align*}
    x'_1 &= \rho_{11}(t) x_1 + \cdots + \rho_{1n}(t) x_n + g_1(t) \\
    \vdots &= \vdots \\
    x'_n &= \rho_{n1}(t) x_1 + \cdots + \rho_{nn}(t) x_n + g_n(t)
\end{align*}
\]

We can write it in matrix form

\[
\vec{x}' = \begin{bmatrix} \rho_{11}(t) & \cdots & \rho_{1n}(t) \\ \vdots & \ddots & \vdots \\ \rho_{n1}(t) & \cdots & \rho_{nn}(t) \end{bmatrix} \vec{x} + \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}
\]

The corresponding homogeneous system is

\[
\vec{x}' = \begin{bmatrix} \rho_{11}(t) & \cdots & \rho_{1n}(t) \\ \vdots & \ddots & \vdots \\ \rho_{n1}(t) & \cdots & \rho_{nn}(t) \end{bmatrix} \vec{x}.
\]

**Principle of Superposition** If the vector functions \( x^{(1)}(t), \ldots, x^{(n)}(t) \) are solutions of the homogeneous system, then

\[
\sum_{i=1}^{n} c_i \vec{x}^{(i)}(t) = \begin{bmatrix} c_1 \vec{x}^{(1)}(t) + \cdots + c_n \vec{x}^{(n)}(t) \end{bmatrix}
\]

is also a solution for any constants \( c_i \).

The Wronskian of these \( n \) functions are

\[
W(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) = \det \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \end{bmatrix}
\]

We say the vector functions \( x^{(1)}(t), \ldots, x^{(n)}(t) \) are solutions form a **fundamental set of solutions** if

- they are linearly independent at each point
- the Wronskian is not zero.

In this case, each solution \( \vec{x}(t) \) of the homogeneous system can be express as

\[
\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \cdots + c_n \vec{x}^{(n)}(t)
\]

If \( x_p(t) \) is a particular solution of the nonhomogeneous system, the general solution is

\[
\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + \cdots + c_n \vec{x}^{(n)}(t) + \vec{x}_p(t)
\]

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