MA 266 Lecture 5

Section 2.2 Separable Equations (contd)

Example 1. Consider the initial value problem

\[ y' = ty(4 - y)/3, \quad y(0) = y_0 > 0. \]

(a) Determine how the behavior of the solution as \( t \) increases depends on the initial value \( y_0 \).

(b) Suppose that \( y_0 = 0.5 \). Find the time \( T \) at which the solution first reaches the value 3.98.

\[
\frac{1}{y(4-y)} \, dy = \frac{t}{3} \, dt \Rightarrow \frac{1}{4} \left( \frac{1}{y} - \frac{1}{y-4} \right) \, dy = d\left( \frac{t^2}{6} \right)
\]

\[
\frac{1}{4} \left( \ln(y) - \ln(y-4) \right) \, dy = d\left( \frac{t^2}{6} \right)
\]

\[
\frac{1}{4} \ln\left| \frac{y}{y-4} \right| = \frac{t^2}{6} + c
\]

\[ y(0) = y_0 \Rightarrow c_0 = \frac{1}{4} \ln\left| \frac{y_0}{y_0-4} \right| - \frac{0}{6} \]

\[
\Rightarrow \frac{1}{4} \ln\left| \frac{y(y_0-4)}{y_0(y-4)} \right| - \frac{t^2}{6} = 0
\]

\( t \to \infty, \ y \to y_0 \)

\[ y_0 = 0.5, \quad y = 3.98 \]

\[
\Rightarrow \frac{1}{4} \ln\left| \frac{3.98 \cdot (-3.5)}{0.5 \cdot (-0.02)} \right| = \frac{1}{6} \frac{2}{
\]

\[ T = \frac{2}{\sqrt{\frac{3}{2} \ln(3.93)}} \]
Homogeneous Equation

Consider the differential equation \( \frac{dy}{dx} = f(x, y) \). If the right hand side

then the equation is said to be \underline{homogeneous}. Such equation can be transformed into \underline{homogeneous} by a change of variable.

Example 2. Solve the differential equation

\[
\frac{dy}{dx} = \frac{y - 4x}{x - y}.
\]
Homogeneous Equation

Consider the differential equation $\frac{dy}{dx} = f(x, y)$. If the right hand side can be expressed as a function of $\frac{y}{x}$ only, then the equation is said to be homogeneous. Such equation can be transformed into separable equation by a change of variable.

Example 2. Solve the differential equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}.$$

Let $u = \frac{y}{x}$, then $\frac{dy}{dx} = \frac{d(ux)}{dx} = \frac{du}{dx} \cdot x + u$

$$\frac{du}{dx} = \frac{\frac{y}{x} = \frac{1}{1 - u}}{1 - u} \cdot x + u = \frac{u - 4}{1 - u}$$

$$\frac{du}{dx} = \frac{(\frac{u^2 - 4}{1 - u})}{x}$$

$$\int \frac{(1 - u)}{(u^2 - 4)} \, du = d \int \ln |x|$$

$$\left[ -\frac{1}{u + 2} - \frac{1}{4} \left( \frac{1}{u + 2} - \frac{1}{u - 2} \right) \right] \, du = d \left( \ln |x| \right)$$

$$- \left[ \frac{3}{4} \left( \frac{1}{u + 2} + \frac{1}{4} \frac{1}{u - 2} \right) \right] du = d \left( \ln |x| \right)$$

$$\frac{3}{8} \ln |u + 2| - \frac{1}{8} \ln |u - 2| = c \left( \ln |x| \right)$$

$$\frac{(u + 2)^3}{(u - 2)^5} = |x|$$

$$\frac{y}{x} = \frac{1}{x}$$

$$\frac{y - 2}{(u + 2)} \Rightarrow |x|$$