Homework 1

Covers Lectures 1 (../lectures/lec_01.ipynb), 2 (../lectures/lec_02.ipynb), and 3 (../lectures/lec_02.ipynb). To solve the following excercises use:

- Common sense
- The obvious, the product, and the sum rules of probability.

1. This exercise demonstrates the probability theory is actually an extension of logic. Assume that you know that "A implies B". That is, your prior information is:
$$I = \{A \implies B\}.$$
Show that:
   A. $p(AB|I) = p(A|I)$ (use common sense).
   B. If $p(A|I) = 1$, then $p(B|I) = 1$.
   C. If $p(B|I) = 0$, then $p(A|I) = 0$.
   D. B and C show that probability theory is consistent with Aristotelian logic. Now, you will discover how it extends it. Show that if B is true, then A becomes more plausible, i.e.
$$p(A|BI) \geq p(A|I).$$
   E. Give at least two examples of D that apply to various scientific fields. To get you started, here are two examples:
      a. A: It is raining. B: There are clouds in the sky. Clearly, $A \implies B$. D tells us that if there are clouds in the sky, raining becomes more plausible.
      b. A: General relativity. B: Light is deflected in the presence of massive bodies. Here $A \implies B$. Observing that B is true makes A more plausible.
   F. Show that if A is false, then B becomes less plausible.
$$P(B|\neg AI) \leq p(B|I).$$
   G. Can you think of an example of scientific reasoning that involves F? For example:
      a. A: It is raining. B: There are clouds in the sky. F tells us that if it is not raining, then it is less plausible that there are clouds in the sky.
   H. Do D and F contradict Karl Popper's *principle of falsification*, "A theory in the empirical sciences can never be proven, but it can be falsified, meaning that it can and should be scrutinized by decisive experiments." Source: Wikipedia (https://en.wikipedia.org/wiki/Karl_Popper))
2. Consider the medical diagnosis example of Lecture 2 (../lectures/lec_02.ipynb).
   A. Compute the probability that a patient that tested negative has tuberculosis. Does the test change our prior state of knowledge about about the patient?
   B. What would a good test look like? Find values for

$p(A|B, I) = p(\text{test is positive}|\text{has tuberculosis}, I)$ and
$p(A|\neg B, I) = p(\text{test is positive}|\text{does not have tuberculosis}, I)$, so that
$p(B|A, I) = p(\text{has tuberculosis}|\text{test is positive}, I) = 0.99$. There are more than one solutions. How would you pick a good one? Thinking in this way can help you set goals if you work in R&D. If you have time, try to figure out whether or not there exists such an accurate test for tuberculosis.

3. Let A and B be independent conditional on I. Prove that:
$$A \perp B|I \iff p(AB|I) = p(A|I)p(B|I).$$
Hint: Use the fact that $A \perp B|I$ *means* $p(A|B, I) = p(A|I)$ (or $p(B|A, I) = p(B|I)$).
4. Let X be a continuous random variable and $F(x) = p(X \le x)$ be its cummulative distribution function. Using only the basic rules of probability, prove that:
   A. The CDF starts at zero and goes up to one:
   $$F(-\infty) = 0 \text{ and } F(+\infty) = 1.$$
   B. $F(x)$ is an increasing function of x, i.e.,
   $$x_1 \le x_2 \implies F(x_1) \le F(x_2).$$
   C. The probability of X being in the interval $[x_1, x_2]$ is:
   $$p(x_1 \le X \le x_2|I) = F(x_2) - F(x_1).$$
5. Let X be a random variable. Prove that:
$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$
6. Let X be the following random variable:
$$X = \begin{cases} 1, & \theta, \\ 0, & 1-\theta, \end{cases}$$
Compute its variance $\mathbb{V}[X]$.
7. Let $X|a, b \sim \text{Beta}(X|a, b)$. Use the properties of the Beta function (https://en.wikipedia.org/wiki/Beta_function) to compute by integration $\mathbb{E}[X|a, b]$ and $\mathbb{V}[X|a, b]$.
8. Let's say that you wish to analyze a coin flipping experiment and you do not know what the probability of getting heads, θ, is. Therefore, you have to model θ as a random variable and assign a probability density to it. Since θ takes values between 0 and 1, you decide to assign a Beta distribution to it. Using the corresponding interactive tool of Lecture 3 (../lectures/lec_03.ipynb), try to find parameters a and b that describe the following states of knowledge:

   - The coin is fair.
   - The coin is slightly biased towards heads.
   - The coin is slightly biased towards tails.
   - The coin is definitely biased towards heads.
   - The coin is definitely biased towards tails.
   - The coin is definitely biased, but I don't know how.

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