MA1200 Exercise for Chapter 7 Techniques of Differentiation
Brief Solutions

First Principle

1. a) $\dfrac{17}{(3x+4)^2}$

b) $$\lim_{h\to 0}\frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}\times\frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}}$$
$$=\lim_{h\to 0}\frac{2h}{h[\sqrt{2(x+h)+1}+\sqrt{2x+1}]}=\frac{2}{2\sqrt{2x+1}}=\frac{1}{\sqrt{2x+1}}=f'(x)$$

Product/Quotient/Chain Rules

2. a) $28x^3-12x+1+\dfrac{1}{x^2}-\dfrac{6}{x^3}-\dfrac{6}{x^4}$

b) $\dfrac{2\sqrt[3]{3}}{3}x^{\frac{-1}{3}}$

c) $\dfrac{-x}{\sqrt{25-x^2}}$

d) $\dfrac{-12}{(3+2x)^2}$

e) $\dfrac{8x-x^3}{(4-x^2)^{1.5}}$

f) $3000(1+2x-5x^2)^{2999}(2-10x)$

g) $2(x^2+4)(2x)(2x^3-1)^3+(x^2+4)^2 3(2x^3-1)^2(6x^2)$

h) $6\sin^2(2x)\cos(2x)+15x^2\sin(x^3+1)$

i) $\dfrac{1}{x\ln x\ln(\ln x)}$

j) $\ln x$

k) $\cot x$

l) $-\tan x$

m) $-1+\dfrac{1}{x}$

n) $-e^x\sin(e^x)$

o) $e^x(x^3\cos x+3x^2\cos x-x^3\sin x)$

p) $\dfrac{(3x^2-5x)e^x(2x+\sin x+2+\cos x)-e^x(2x+\sin x)(6x-5)}{(3x^2-5x)^2}$

q) $\dfrac{(n-1)x^n-nx^{n-1}+1}{(x-1)^2}$

r) $\dfrac{-x^2-4x+6}{(x^2-3x)^2}$

s) $2\cos(2x)\cos(3x)-3\sin(2x)\sin(3x)$

t) $\cos(\ln[\cos(\ln(2x)+1)]+1)\dfrac{-\sin(\ln(2x)+1)}{\cos(\ln(2x)+1)}\dfrac{1}{x}$

3. a) $y'=x/y$

b) $y'=-\dfrac{2x+y}{x+2y}$

c) $\dfrac{(x^2+y^2)x-y}{x-(x^2+y^2)y}$

4. a) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9t^2}{4t} = \frac{9}{4}t = \frac{9}{4}$ when $t = 1$. Tangent: $y-5=\frac{9}{4}(x-3)$, $y = 2.25x - 1.75$ Normal:

$y-5=\frac{-4}{9}(x-3)$, $y=\frac{-4}{9}x+\frac{19}{3}$

b) $\frac{dy}{dx} = 2\sqrt{t} + \frac{1}{t} = \frac{17}{4}$ when $t = 4$. Tangent: $y = \frac{17}{4}x - 5$,

Normal: $y = -\frac{4}{17}x + \frac{135}{34}$

5. a) $(x+1)^{\cot x}\left[-\csc^2 x\ln(x+1)+\frac{\cot x}{x+1}\right]$

b) $(x^4+2x^2)^{e^x}\left[e^x\ln(x^4+2x^2)+\frac{e^x(4x^3+4x)}{x^4+2x^2}\right]$

c) $(\sin 5x)^{x^2+2}[2x\ln(\sin 5x)+5(x^2+2)\cot 5x]+3$

6. a) $y^{(n)} = 3(4^n)e^{4x}$

b) $y^{(n)} = 5(6^n)\sin\left(6x-7+\frac{n\pi}{2}\right)$

c) $y^{(n)} = 2^{n+1}(-1)^n n!(2x-1)^{-n-1}$

d) $$y^{(n)} = \begin{cases} \frac{11!}{(11-n)!}x^{11-n} - 6\frac{3!}{(3-n)!}x^{3-n} & 1 \le n \le 3 \\ \frac{11!}{(11-n)!}x^{11-n} & \text{when } 4 \le n \le 11 \\ 0 & 12 \le n \end{cases}$$

Leibniz's rule

7. a) $$y^{(n)} = \begin{cases} e^{2x}x^2(3+2x) & n=1 \\ 2e^{2x}x(2x^2+6x+3) & \text{when } n=2 \\ 2^{n-3}e^{2x}[2^3x^3+3(2^2)nx^2+3(2)n(n-1)x+n(n-1)(n-2)] & n\ge 3 \end{cases}$$

b) $$y^{(n)} = \begin{cases} (2x-4)\cos(2x-1)+2(x^2-4x+7)\cos\left(2x-1+\frac{\pi}{2}\right) & \text{when } n=1 \\ 2^n\cos\left(2x-1+\frac{n\pi}{2}\right)(x^2-4x+7) & \\ \quad +n2^{n-1}\cos\left(2x-1+\frac{(n-1)\pi}{2}\right)(2x-4) & \\ \quad +n(n-1)2^{n-2}\cos\left(2x-1+\frac{(n-2)\pi}{2}\right) & \text{when } n\ge 2 \end{cases}$$

c) $y^{(n)}=\begin{cases}(1+x)^{-2}x(2+x) & \text{when } n=1\\ (-1)^n n!(1+x)^{-n-1} & \text{when } n\geq 2\end{cases}$

Miscellaneous

8. If $y=x\sin x$, prove that $x^2y''-2xy'+\left(2+x^2\right)y=0$.

Proof:

$y=x\sin x\Rightarrow y'=\sin x+x\cos x\Rightarrow y''=2\cos x-x\sin x$. Hence $x^2y''-2xy'+(2+x^2)y=\cdots=0$.

9. If $u=\sqrt{ax^2+2bx+c}$, prove that $\frac{d}{dx}(xu)=\frac{2ax^2+3bx+c}{u}$.

Proof:

$\frac{du}{dx}=\frac{d}{dx}\left((ax^2+2bx+c)^{\frac{1}{2}}\right)=\frac{ax+b}{(ax^2+2bx+c)^{\frac{1}{2}}}=\frac{ax+b}{u}$. Then $\frac{d}{dx}(xu)=x\frac{ax+b}{u}+u=\cdots=\frac{2ax^2+3bx+c}{u}$.

*10. Find the values of *a* and *b* (in terms of *c*) provided $f'(c)$ exists, where $f(x)=\begin{cases}\frac{1}{|x|} & \text{if } |x|>c\\ ax+b & \text{if } |x|\leq c\end{cases}$.

Solution:

$$f(x)=\begin{cases}\frac{1}{|x|} & \text{if } |x|>c\\ ax+b & \text{if } |x|\leq c\end{cases}\Rightarrow f(x)=\begin{cases}\frac{1}{x} & \text{if } x>c\\ ax+b & \text{if } -c\leq x\leq c\\ \frac{1}{-x} & \text{if } x<-c\end{cases}$$

In order that $f'(c)$ exists, $f(x)$ must be continuous at $x=c$.

That is, $\lim_{x\to c^+}f(x)=\lim_{x\to c^+}\frac{1}{x}=\frac{1}{c}=\lim_{x\to c^-}f(x)=ac+b=f(c)$.

$$\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c^+}\frac{\frac{1}{x}-(ac+b)}{x-c}=\lim_{x\to c^+}\frac{\frac{1}{x}-\frac{1}{c}}{x-c}=\lim_{x\to c^+}\frac{\frac{c-x}{xc}}{x-c}=-\lim_{x\to c^+}\frac{\frac{x-c}{xc}}{x-c}=-\lim_{x\to c^+}\frac{1}{xc}=-\frac{1}{c^2}.$$

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c^-}\frac{ax+b-(ac+b)}{x-c}=\lim_{x\to c^-}\frac{a(x-c)}{x-c}=a.$$

That $f'(c)$ exists means $\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}=-\frac{1}{c^2}=\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}=a$.

Therefore, $a=-\frac{1}{c^2}$. Putting in $ac+b=\frac{1}{c}$, we have $-\frac{1}{c}+b=\frac{1}{c}\Rightarrow b=\frac{2}{c}$. Hence $\begin{cases}a=-\frac{1}{c^2}\\ b=\frac{2}{c}\end{cases}$.

11. A function y of x is defined by the equation $\sin(x-y)=m\sin y$. Express y explicitly in terms of x.

Hence, or otherwise, show that $\dfrac{dy}{dx}=\dfrac{1+m\cos x}{1+2\,m\ \cos x+m^2}$.

<u>Proof</u>:

$$\sin(x-y)=m\sin y \Rightarrow \sin x\cos y-\cos x\sin y=m\sin y \Rightarrow \tan y=\frac{\sin x}{m+\cos x} \Rightarrow y=\tan^{-1}\left(\frac{\sin x}{m+\cos x}\right)$$

Then

$$\frac{dy}{dx}=\frac{d\left[\tan^{-1}\left(\dfrac{\sin x}{m+\cos x}\right)\right]}{dx}=\cdots=\frac{1+m\cos x}{1+2m\cos x+m^2}$$

12. Let $y=\left(x+1\right)^{\cot x}$, find $\dfrac{dy}{dx}$.

<u>Solution</u>:

$$\frac{dy}{dx}=-\left(x+1\right)^{\cot x}\csc^2 x\ln\left(x+1\right)+\frac{\cot x}{x+1}\left(x+1\right)^{\cot x}$$

13. Show that $f'(x)=0$, where $f(x)=\tan^{-1}x+\tan^{-1}\dfrac{1}{x}$.

<u>Solution</u>:

$$f'(x)=\frac{1}{1+x^2}-\frac{x^2}{1+x^2}\frac{1}{x^2}=0.$$

14. Consider the parametric curve $\begin{cases} x=t^2+2 \\ y=t^3 \end{cases}$ where $-\infty<t<\infty$.

Find $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ and the equation of the tangent line to the curve at the point (3, 1).

<u>Solution</u>:

$$\frac{dy}{dx}=\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \Rightarrow \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}\right)=\frac{d}{dt}\left(\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}\right)\frac{dt}{dx}=\left[\frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^2}\right]\frac{dt}{dx}$$

The slope of the tangent line to the curve at the point $\left(3,1\right)=\dfrac{dy}{dx}\Big|_{x=3}=\dfrac{3}{2}$.

Therefore, the equation of the tangent line to the curve at the point $\left(3,1\right)$ is: $y-1=\dfrac{3}{2}\left(x-3\right)$.

15. By Leibnitz's theorem on repeated differentiation, find the nth derivatives of the function $y=e^x\cos 2x$.

Solution:

By Leibnitz's theorem on repeated differentiation, we have

$\frac{d^n}{dx^n}(fg)=\sum_{k=0}^{n}\binom{n}{k}\frac{d^k g}{dx^k}\frac{d^{n-k} f}{dx^{n-k}}$, where $\frac{d^0 f}{dx^0}=f$, $\frac{d^0 g}{dx^0}=g$, $\binom{n}{k}=\frac{n!}{k!(n-k)!}$, $k=0,1,\cdots,n$ and $0!=1$.

We have $\frac{d^n}{dx^n}\left[e^x\cos(2x)\right]=\sum_{k=0}^{n}2^k\binom{n}{k}e^x\cos\left(2x+\frac{n\pi}{2}\right)$

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