Supplementary Problems

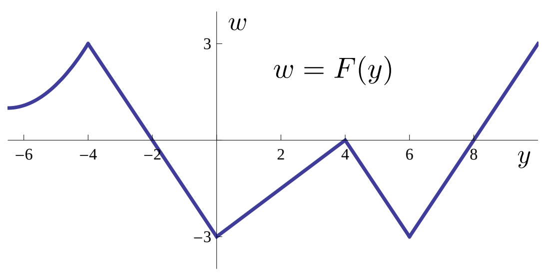
A. For what value(s) of A, if any, will $y = Ate^{-2t}$ be a solution of the differential equation $2y' + 4y = 3e^{-2t}$? For what value(s) of B, if any, will $y = Be^{-2t}$ be a solution?

B. Using the substitution $u(x) = y + x$, solve the differential equation $\dfrac{dy}{dx} = (y + x)^2$.

C. Using the substitution $u(x) = y^3$, solve the differential equation $y^2\dfrac{dy}{dx} + \dfrac{y^3}{x} = \dfrac{2}{x^2}$ $(x > 0)$.

D. Find the explicit solution of the Separable Equation $\dfrac{dy}{dt} = y^2 - 4y$, $y(0) = 8$. What is the largest open interval containing $t = 0$ for which the solution is defined?

E. The graph of $F(y)$ vs y is as shown:



(a) Find the equilibrium solutions of the autonomous differential equation $\dfrac{dy}{dt} = F(y)$.

(b) Determine the stability of each equilibrium solution.

F. Solve the differential equation $\dfrac{dw}{dt} = \dfrac{2tw}{w^2 - t^2}$

G. (a) If $y' = -2y + e^{-t}$, $y(0) = 1$ then compute $y(1)$.

(b) Experiment using the Euler Method (`eul`) with step sizes of the form $h = 1/n$ to find the smallest integer n which will give a value y_n that approximates the above true solution at $t = 1$ within 0.05.

H. (a) If $y' = 2y - 3e^{-t}$, $y(0) = 1$ then compute $y(1)$.

(b) Experiment using the Euler Method (`eul`) with step sizes of the form $h = 1/n$ to find the smallest integer n which will give a value y_n that approximates the above true solution at $t = 1$ within 0.05.

I. Approximation methods for differential equations can be used to estimate definite integrals:

(a) Show that $y(t) = \displaystyle\int_0^t e^{-u^2}\,du$ satisfies the initial value problem $\dfrac{dy}{dt} = e^{-t^2}$, $y(0) = 0$.

(b) Use the Euler Method (`eul`) with $h = 1/2$ to approximate the integral $\int_0^2 e^{-u^2}\,du$.

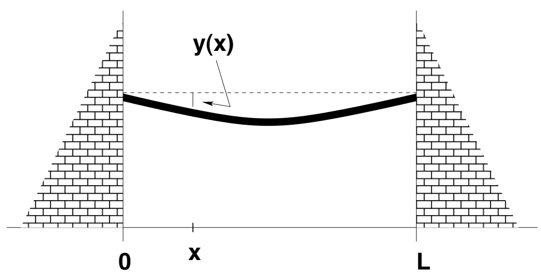
J. Given that the general solution to $t^2y'' - 4ty' + 4y = 0$ is $y = C_1t + C_2t^4$, solve the following initial value problem:

$$\begin{cases} t^2y'' - 4ty' + 4y = -2t^2 \\ y(1) = 2,\ y'(1) = 0 \end{cases}$$

K. From the theory of elasticity, if the ends of a horizontal beam (of uniform cross-section and constant density) are supported at the same height in vertical walls, then its vertical displacement $y(x)$ satisfies the Boundary Value Problem

$$\begin{cases} y^{(4)} = -P \\ y(0) = y(L) = 0 \\ y'(0) = y'(L) = 0 \end{cases}$$

where $P > 0$ is a constant depending on the beam's density and rigidity and L is the distance between supporting walls:



(a) Solve the above boundary value problem when $L = 4$ and $P = 24$.

(b) Show that the maximum displacement occurs at the center of the beam $x = \frac{L}{2} = 2$.

L. Laplace transforms may be used to find particular solutions to some nonhomogeneous differential equations. Use Laplace transforms to find a particular solution, $y_p(t)$, of $y'' + 4y = 20e^t$.

Hint: Solve the initial value problem $\begin{cases} y'' + 4y = 20e^t \\ y(0) = 0,\ y'(0) = 0 \end{cases}$

M. Tank 1 initially contains 50 gals of water with 10 oz of salt in it, while Tank 2 initially contains 20 gals of water with 15 oz of salt in it. Water containing 2 oz/gal of salt flows into Tank 1 at a rate of 5 gal/min and the well-stirred mixture flows from Tank 1 into Tank 2 at the same rate of 5 gal/min. The solution in Tank 2 flows out to the ground at a rate of 5 gal/min. If $x_1(t)$ and $x_2(t)$ represent the number of ounces of salt in Tank 1 and Tank 2, respectively, **set up but do not solve** an initial value problem describing this system.

N. If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent solutions to the 2×2 system $\mathbf{x}' = A\mathbf{x}$, then the matrix $\Phi(t) = (\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t))$ is called a *Fundamental Matrix* for the system. Find a Fundamental Matrix $\Phi(t)$ of the system $\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}$.

O. Laplace transforms may be used to find solutions to some linear systems of differential equations. Consider the linear system of differential equations:

$$\begin{cases} x' = x + y \\ y' = 4x + y \end{cases} \tag{$*$}$$

with initial conditions $x(0) = 0$ and $y(0) = 2$.

(a) Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transforms of the functions $x(t)$ and $y(t)$, respectively. Take the Laplace transform of each of the differential equations in $(*)$ and solve for $X(s)$ (i.e., eliminate $Y(s)$).

(b) Using the function $X(s)$ from (a), determine $x(t)$.

(c) Use the expression for $x(t)$ and the first equation in $(*)$ to determine $y(t)$.

P. Find a particular solution $\mathbf{x}_p(t)$ of these nonhomogeneous systems:

(a) $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5e^{2t} \\ 3 \end{pmatrix}$

(b) $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10\cos t \\ 0 \end{pmatrix}$