MATH 266 Quiz 1

Name: Solution PID:

Solve the problem systematically and show all your work.

1. According to Newton's law of cooling, the temperature $u(t)$ of an object satisfies the differential equation

$$du/dt = -k(u - T),$$

where T is the constant ambient temperature and k is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$.

(3pts)(a) Find the temperature of the object at any time.

(2pts)(b) Let τ be the time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between τ and k.

Sol: (a) Solve I.V.P

$$\frac{du}{dt} = -k(u - T)$$

$$u(0) = u_0$$

(1) $u = T$ is an equilibrium solution

(2) $u \neq T$ $\quad \frac{du/dt}{u-T} = -k$

$$\Rightarrow \ln|u-T| = -kt + C_1$$

$$\Rightarrow u = T \pm e^{C_1} e^{-kt}$$

Combine (1) & (2)

$$u = T + Ce^{-kt}$$

where C is an arbitrary constant

$$u(0) = u_0$$

$$\Rightarrow u_0 = T + Ce^{-k(0)} = T + C$$

$$\Rightarrow C = u_0 - T$$

Thus

$$\boxed{u(t) = T + (u_0 - T)e^{-kt}}$$

(b) Solve time τ when

$$\left.\begin{array}{l} u - T = \frac{1}{2}(u_0 - T) \\ u - T = (u_0 - T)e^{-kt} \end{array}\right\}$$

$$\Rightarrow \frac{1}{2} = e^{-k\tau}$$

$$\Rightarrow \ln\frac{1}{2} = -k\tau$$

$$\Rightarrow k\tau = \ln 2$$

or $\boxed{\tau = \frac{\ln 2}{k}}$

(5pts)2. Solve the initial value problem

$$ty' + 2y = t^2 - t + 1,$$
$$y(1) = \frac{1}{2},$$

for $t > 0$.

Sol: $t > 0$ we have

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

use the method of integrating factors

$$p(t) = \frac{2}{t} \qquad g(t) = t - 1 + \frac{1}{t}$$

$$I(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln t^2} = t^2$$

$$y = \frac{1}{t^2}\left[\int (t - 1 + \tfrac{1}{t})t^2\, dt + C\right]$$

$$\boxed{y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{C}{t^2}}$$

$$y(1) = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C$$

$$\Rightarrow C = \frac{1}{12}$$

Then the solution to the I.V.P

$$\boxed{y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}}$$