MATH 266 Quiz 2

Name: Solution PID:

Solve the problem systematically and show all your work.

(2pts)1. Without solving the equation, find the largest interval in which the solution of the initial value problem

$$(t-2)\frac{dy}{dx} - 5(\cos t)y = \ln(6-t), \qquad y(3) = 2$$

is guaranteed to exist by the Existence and Uniqueness Theorem.

Sol: divide the equation both sides by $t-2$, we have

$$\frac{dy}{dx} - \frac{5\cos t}{t-2}y = \frac{\ln(6-t)}{t-2} \qquad \text{(linear equation!)}$$

use the 1st Thm in section 2.4, The largest interval is the interval for which $p(t) = -\frac{5\cos t}{t-2}$ and $g(t) = \frac{\ln(6-t)}{t-2}$ are continuous and includes the initial value $t_0 = 3$

$p(t)$ & $g(t)$ are continuous on $t<6$ & $t \neq 2$ $\Leftrightarrow$ $(2,6)$ & $(-\infty, 2)$; $t_0 = 3 \in (2,6)$ $\Rightarrow$ The largest interval is $\boxed{(2,6)}$

(2pts)2. Find all the equilibrium solutions to the differential equation

$$\frac{dy}{dx} = (1-y^2)(y^2-4)$$

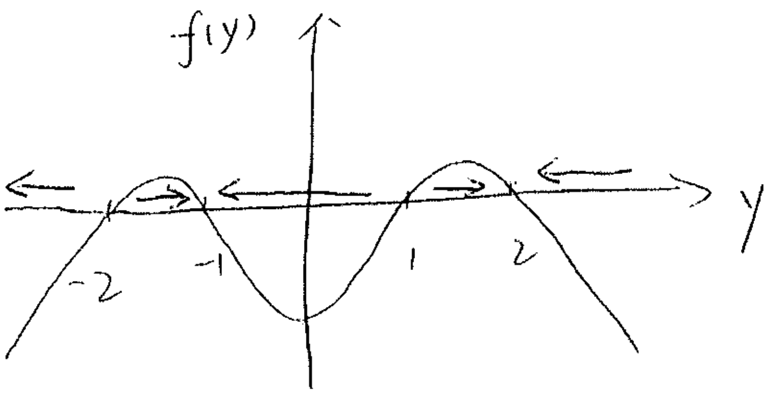
and use the phase line to identify each as asymptotically stable, unstable or semistable.

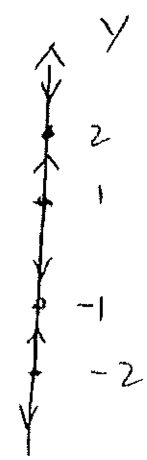
Sol: $f(y) = (1-y^2)(y^2-4) = (1-y)(1+y)(y-2)(y+2) = 0$

$\Rightarrow$ equilibrium solutions are $y = -2, -1, 1$ and 2

Sketch the graph of $f(y)$ versus y

phase line





Checking the arrows

$y = -1, 2$ are asymptotically stable

$y = -2, 1$ are unstable

(3pts)3. Solve the differential equation

$$2xydx + (x^2+1)dy = 0.$$

Sol: $M = 2xy \Rightarrow \frac{\partial M}{\partial y} = 2x$

$N = x^2+1 \Rightarrow \frac{\partial N}{\partial x} = 2x$ $\Bigg\} \Rightarrow$ D.E is exact

$\frac{\partial \phi}{\partial x} = M = 2xy \Rightarrow \phi = x^2y + h(y) \Rightarrow \frac{\partial \phi}{\partial y} = x^2 + h'(y)$

$\frac{\partial \phi}{\partial y} = N = x^2+1$ $\Bigg\} \Rightarrow h(y) = y$

$\phi(x,y) = x^2y + y \Rightarrow$ general solution to the D.E is $\boxed{x^2y + y = c}$

(3pts)4. A tank initially contains 10L of water in which there is 20g of salt dissolved. A solution containing 4g/L of salt is pumped into the container at a rate of 2 L/min, and the well-stirred mixture runs out at a rate of 1 L/min. What is the concentration in the tank after 10 minutes?

Sol: $V_0 = 10L$ $A_0 = 20g$ $C_1 = 4g/L$ $r_1 = 2L/min$ $r_2 = 1L/min$

Need to find $C_2(10) = \frac{A(10)}{V(10)}$ b/c $C_2(t) = \frac{A(t)}{V(t)}$

$\frac{dV}{dt} = r_1 - r_2 = 1 \Rightarrow V = t + c$

$V(0) = V_0 = 10$ $\Bigg\} \Rightarrow V = t + 10 \Rightarrow V(10) = 20$

$\frac{dA}{dt} = r_1C_1 - r_2C_2 = 8 - 1 \times \frac{A}{V} \Rightarrow \frac{dA}{dt} + \frac{A}{t+10} = 8$

$I(t) = e^{\int \frac{1}{t+10} dt} = t + 10$

$\frac{d}{dt}(A(t+10)) = 8(t+10) \Rightarrow A = \frac{1}{t+10}[4t^2 + 80t + c]$

$A(0) = A_0 = 20$ $\Bigg\} \Rightarrow c = 200$

$\Rightarrow A = \frac{4t^2 + 80t + 200}{t+10} \Rightarrow A(10) = \frac{400 + 800 + 200}{20} = 70$

Thus $C_2(10) = \frac{A(10)}{V(10)} = \frac{70}{20} = \boxed{3.5\ g/L}$