i.e., the normal velocity on the boundary is proportional to the excess pressure on the boundary. The coefficient $\chi$ is called the acoustic impedance of the obstacle $D$, and is, in general, a space dependent function defined on the boundary $\partial D$. This impedance condition leads to a boundary-value problem for the velocity potential $u$ of the form

$$\partial_n u + i\lambda u = 0,$$

where $\lambda = \chi \rho (\omega + i\gamma)$.

### 4.2 Green’s representation theorem

We begin our analysis by establishing the basic property that any solution to the Helmholtz equation can be represented as the combination of a single- and a double-layer acoustic surface potential. It is easily verified that the function

$$G(x, y) = \frac{1}{4\pi} e^{i\kappa |x-y|}, \quad x, y \in \mathbb{R}^3, \quad x \neq y,$$

is a solution to the Helmholtz equation

$$\Delta G(x, y) + \kappa^2 G(x, y) = 0$$

with respect to $x$ for any fixed $y$. Because of its polelike singularity at $x = y$, the function $G$ is called a fundamental solution to the Helmholtz equation, i.e.,

$$\Delta G(x, y) + \kappa^2 G(x, y) = -\delta(x - y).$$

Given a function $\phi \in C(\partial D)$, the function

$$u(x) = \int_{\partial D} G(x, y) \phi(y) ds_y, \quad x \in \mathbb{R}^2 \setminus \partial D,$$

is called the acoustic single-layer potential with density $\phi$. Since for $x \in \mathbb{R}^3 \setminus \partial D$, we can differentiate under the integral sign, we see that $u$ is a solution of the Helmholtz equation.

Given a function $\psi \in C(\partial D)$, the function

$$v(x) = \int_{\partial D} \partial_{n_y} G(x, y) \psi(y) ds_y, \quad x \in \mathbb{R}^3 \setminus \partial D$$

is called the acoustic double-layer potential with density $\psi$. We assume unit normal $n$ to be directed into the exterior domain $\mathbb{R}^3 \setminus \overline{D}$. We note that the double-layer
potential $v$ is also a solution to the Helmholtz equation. In the following, we shall distinguish by indices + and − the limits obtained by approaching the boundary $\partial D$ from inside $\mathbb{R}^3 \setminus \overline{D}$ and $D$, respectively, i.e.,

$$v_+(x) = \lim_{y \to x, \ y \in \mathbb{R}^3 \setminus \overline{D}} v(y), \quad v_-(x) = \lim_{y \to x, \ y \in D} v(y), \quad x \in \partial D.$$ 

For any domain $\Omega$ with boundary $\partial \Omega$ of class $C^2$, we introduce the linear space $X(G)$ of all complex-valued function $u \in C^2(G) \cap C(\overline{G})$ for which the normal derivative on the boundary exists in the sense that the limit

$$\partial_n u(x) = \lim_{h \to 0, \ h > 0} \langle n(x), \nabla u(x - h n(x)) \rangle, \quad x \in \partial G,$$

exists uniformly on $\partial G$. We note that the assumption $u, v \in X(G)$ suffices to guarantee the validity of the first Green’s theorem

$$(4.3) \quad \int_G u \Delta v dx = \int_{\partial G} u \partial_n v ds - \int_G \nabla u \cdot \nabla v dx$$

and the second Green’s theorem

$$(4.4) \quad \int_G (u \Delta v - v \Delta u) dx = \int_{\partial G} (u \partial_n v - v \partial_n u) ds$$

for a bounded domain $G$ with $C^2$ boundary $\partial G$.

We denote by $D$ a bounded region in $\mathbb{R}^3$ with a boundary $\partial D$ consisting of a finite number of disjoint, closed, bounded surfaces belonging to the class $C^2$. The exterior $\mathbb{R}^3 \setminus D$ is assumed to the connected, whereas $D$ itself may have more than one component. We assume the normal $n$ to $\partial D$ to be directed into the exterior of $D$.

**Theorem 4.2.1.** Let $u \in X(D)$ be a solution to the Helmholtz equation

$$\Delta u + \kappa^2 u = 0 \quad \text{in } D.$$

Then

$$\int_{\partial D} \left[ u(y) \partial_n y G(x, y) - \partial_n y u(y) G(x, y) \right] ds_y = \begin{cases} -u(x), & x \in D, \\ 0, & x \in \mathbb{R}^3 \setminus \overline{D}. \end{cases}$$

**Proof.** We choose an arbitrary fixed point $x \in D$ and circumscribe it with a sphere $B_{x,r} := \{ y \in \mathbb{R}^3, \ |x - y| = r \}$. We assume the radius $r$ to be small enough such that $B_{x,r} \subset D$ and direct the unit normal $\nu$ to $B_{x,r}$ into the interior of $B_{x,r}$, as seen
4.2. Green’s Representation Theorem

Figure 4.1: Domain $D$ and ball $B_{x,r}$.

in Figure 4.1. Now we apply the second Green’s theorem (4.4) to the function $u(y)$ and $G(x, y)$ in the region $\{y \in D, \ |x - y| > r\}$ to obtain

$$\int_{\partial D + B_{x,r}} \left[ u(y) \frac{\partial G(x, y)}{\partial \nu_y} - \frac{\partial u}{\partial \nu}(y)G(x, y) \right] ds(y) = 0.$$ 

Since on $B_{x,r}$, we have

$$G(x, y) = \frac{e^{i \kappa r}}{4 \pi r}, \quad \nabla_y G(x, y) = \left( \frac{1}{r} - i \kappa \right) \frac{e^{i \kappa r}}{4 \pi r} \nu(y).$$

Simple calculation, using the mean value theorem, shows that

$$\lim_{r \to 0} \int_{B_{x,r}} \left[ u(y) \frac{\partial G(x, y)}{\partial \nu_y} - \frac{\partial u}{\partial \nu}(y)G(x, y) \right] ds(y) = u(x).$$

The statement for $x \in \mathbb{R}^3 \setminus \overline{D}$ readily follows from Green’s theorem applied to the function $u(y)$ and $G(x, y)$ in the region $D$.

$$\square$$

Straightforward calculations show that

$$\left( \frac{x}{|x|}, \nabla_x G(x, y) \right) - i \kappa G(x, y) = O \left( \frac{1}{|x|^2} \right), \quad |x| \to \infty$$

and

$$\left( \frac{x}{|x|}, \nabla_x \frac{\partial G(x, y)}{\partial \nu(y)} \right) - i \kappa \frac{\partial G(x, y)}{\partial \nu(y)} = O \left( \frac{1}{|x|^2} \right), \quad |x| \to \infty$$

uniformly for all directions $x/|x|$ and uniformly for all $y$ contained in the bounded set $\partial D$. From this we conclude the following.
Theorem 4.2.2. Both the single-layer acoustic potential defined by (4.1) and the double-layer acoustic potential defined by (4.2) satisfy the Sommerfeld radiation condition
\[ \left( \frac{x}{|x|}, \nabla u(x) \right) - i \kappa u(x) = o \left( \frac{1}{|x|} \right), \quad |x| \to \infty \]
uniformly for all directions \( x/|x| \).

As we shall soon see, the Sommerfeld radiation condition completely characterizes the behavior of solutions to the Helmholtz equation at infinity.

Theorem 4.2.3. Let \( u \in X(\mathbb{R}^3 \setminus \overline{D}) \) be a solution to the Helmholtz equation
\[ \Delta u + \kappa^2 u = 0 \quad \text{in} \ \mathbb{R}^3 \setminus \overline{D} \]
satisfying the Sommerfeld radiation condition
\[ \left( \frac{x}{|x|}, \nabla u(x) \right) - i \kappa u(x) = o \left( \frac{1}{|x|} \right), \quad |x| \to \infty \]
uniformly for all directions \( x/|x| \). Then
\[ \int_{\partial D} \left[ u(y) \partial_n G(x, y) - \partial_n u(y) G(x, y) \right] ds_y = \begin{cases} 0, & x \in D, \\ u(x), & x \in \mathbb{R}^3 \setminus \overline{D}. \end{cases} \]

Proof. We first show that
\[ \int_{|y|=R} |u|^2 ds = O(1), \quad R \to \infty. \tag{4.5} \]

To accomplish this, we first observe that from the radiation condition it follows that
\[ 0 = \lim_{R \to \infty} \int_{|y|=R} \left| \frac{\partial u}{\partial \nu} - i \kappa u \right|^2 ds \]
\[ = \lim_{R \to \infty} \int_{|y|=R} \left| \frac{\partial u}{\partial \nu} \right|^2 + |\kappa|^2 |u|^2 + 2 \text{Im} \left( \kappa u \frac{\partial \bar{u}}{\partial \nu} \right) ds \tag{4.6} \]
where \( \nu \) denotes the outward unit normal to the sphere \( B_R := \{ y \in \mathbb{R}^3, \ |y| = R \} \).

We take \( R \) large enough so that \( B_R \subset \mathbb{R}^3 \setminus \overline{D} \) and apply the first Green’s theorem in the domain \( D_R := \{ y \in \mathbb{R}^3 \setminus \overline{D}, \ |y| < R \} \) to obtain
\[ \kappa \int_{|y|=R} u \frac{\partial \bar{u}}{\partial \nu} ds = \kappa \int_{\partial D} u \frac{\partial \bar{u}}{\partial \nu} ds - \bar{\kappa} |\kappa|^2 \int_{D_R} |u|^2 dy + \kappa \int_{D_R} |\nabla u|^2 dy. \]
Now we substitute the imaginary part of the above equation into (4.6) and find that
\[
\lim_{R \to \infty} \int_{|y| = R} \left( \frac{\partial u}{\partial \nu} \right)^2 + |\kappa|^2 |u|^2 \, ds + 2\text{Im} \int_{D_R} |\kappa|^2 |u|^2 + |\nabla u|^2 \, dy
\]
\[= -2\text{Im} \kappa \int_{\partial D} u \bar{u} \frac{\partial}{\partial \nu} ds. \quad (4.7)\]

All four terms on the left-hand side of above equation are nonnegative since \(\text{Im} \kappa \geq 0\). Hence these terms must be individually bounded as \(R \to \infty\) since their sum tends to a finite limit. Equation (4.5) follows immediately.

We now note the identity
\[
\int_{|y| = R} \left[ u(y) \frac{\partial G(x, y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu} (y) G(x, y) \right] ds(y)
\]
\[= \int_{|y| = R} u(y) \left[ \frac{\partial G(x, y)}{\partial \nu(y)} - i\kappa G(x, y) \right] ds(y)
\]
\[- \int_{|y| = R} G(x, y) \left[ \frac{\partial u}{\partial \nu} (y) - i\kappa u(y) \right] ds(y) =: I_1 + I_2 \]
and apply Schwartz’s inequality to each of the integrals \(I_1\) and \(I_2\). From the radiation condition
\[
\frac{\partial G(x, y)}{\partial \nu(y)} - i\kappa G(x, y) = O \left( \frac{1}{R^2} \right), \quad y \in B_R,
\]
for the fundamental solution and (4.5) we see that \(I_1 = O(1/R)\) as \(R \to \infty\). The radiation condition and \(G(x, y) = O(1/R), y \in B_R\), yield \(I_2 = o(1)\) for \(R \to \infty\). Hence
\[
\lim_{R \to \infty} \int_{|y| = R} \left[ u(y) \frac{\partial G(x, y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu} (y) G(x, y) \right] ds(y) = 0.
\]
The proof is now completed as in Theorem 4.2.1 by applying the second Green’s theorem in the domain \(\{ y \in D_R, |x - y| > r \}\) if \(x \in \mathbb{R}^3 \setminus \overline{D}\) or \(D_R\) if \(x \in D\). \(\square\)

**Remark 4.2.4.** It is obvious that any solution of the Helmholtz equation satisfying the Somerfeld radiation condition automatically satisfies
\[
u(x) = O \left( \frac{1}{|x|} \right), \quad |x| \to \infty
\]
uniformly for all directions \(x/|x|\).

Note that it is not necessary to impose this additional condition for the representation theorem to be valid. Physically, the fundamental solution \(G(x, y)\) describes an outgoing spherical wave of the form
\[
e^{i(\kappa|x - y| - \omega t)}
\]
\[
\frac{1}{4\pi|x - y|}.
\]