4.3 Boundary integral equations

We introduce the equivalent sources for the Helmholtz equation and establish their connections to the naturally induced sources for the sound-soft, sound-hard, and impedance obstacles.

An equivalent source for a time-harmonic wave u in a domain D is made of monopoles, dipoles, or their combination on the boundary which reproduces the wave in the domain. The problems of determining the equivalent sources given u is referred to as the interior (scattering) problems. There are three standard interior problems for the monopoles, dipoles, and their linear combination.

If the domain is the support of an obstacle, of sound-soft or sound-hard or impedance type, the scattered wave can be expressed as the potential of the single or double or combined layer, respectively. These monopole, dipole, and combined sources are referred to as the naturally induced sources for the soft, hard, and impedance obstacles, respectively. Thus, for instance, double-layer potential for the exterior Neumann problem of the Helmholtz equation exploys the naturally induced (dipole) sources, whereas combined potential for the exterior Dirichlet problem does not.

We will establish connections between these interior and exterior problems by identifying the naturally induced sources for the soft, hard, impedance obstacles with the equivalent sources of monopole, dipole, combined types, respectively.

Let Ω be a domain and $\kappa > 0$ be the wavenumber. Throughout, we work with smooth boundary $\partial\Omega$; we also assume that the incident wave u_0 is generated by sources away from the boundary, so that the classical theory for layer potentials holds and that the scattering solutions are smooth.

Denote by G the fundamental solution of the Helmholtz equation. Let p and q be the single and double layer potentials of smooth density σ

$$p(x) = \int_{\partial\Omega} G(x,\xi)\sigma(\xi)ds(\xi), \tag{4.8}$$

$$q(x) = \int_{\partial\Omega} \frac{\partial G(x,\xi)}{\partial n(\xi)}\sigma(\xi)ds(\xi). \tag{4.9}$$

We have the four well-known jump conditions across the boundary

$$p_+ - p_- = 0, \quad \partial_n q_+ - \partial_n q_- = 0, \tag{4.10}$$

$$q_+ - q_- = \sigma, \quad \partial_n p_+ - \partial_n p_- = -\sigma. \tag{4.11}$$

Lemma 4.3.1. *Let p and q be the single and double layer potentials of a smooth density σ. Let μ, λ be scalars. then*

$$(\mu\partial_n + \mathrm{i}\lambda)(\mu q + \mathrm{i}\lambda p)(x^+) = (\mu\partial_n + \mathrm{i}\lambda)(\mu q + \mathrm{i}\lambda p)(x^-).$$

We introduce the equivalent and naturally induced sources, and use them to establish connections between the interior and exterior scattering problems.

The interior problems: equivalent sources. Let us refer to the source outside Ω, which generate the incident wave u_0, as the primary sources. As is well-known, it is possible to use equivalent sources on $\partial\Omega$ to reproduce the same incident wave u_0 inside Ω; the Green's representation theorem

$$u_0(x) = \int_{\partial\Omega} \left[\partial_n u_0(\xi) G(x,\xi) - u_0(\xi) \frac{\partial G(x,\xi)}{\partial n(\xi)} \right] ds(\xi) \tag{4.12}$$

for example, does it with monopole density $\partial_n u_0$ and dipole density u_0 on $\partial\Omega$.

We introduce three interior problems as to determine the equivalent sources α, β, and γ such that their corresponding single, double, and combined layer potentials all match u_0 inside Ω:

$$u_0(x) = \int_{\partial\Omega} G(x,\xi)\alpha(\xi) ds, \quad x \in \Omega, \tag{4.13}$$

$$u_0(x) = \int_{\partial\Omega} \frac{\partial G(x,\xi)}{\partial n(\xi)} \beta(\xi) ds, \quad x \in \Omega, \tag{4.14}$$

$$u_0(x) = \int_{\partial\Omega} \left[\frac{\partial G(x,\xi)}{\partial n(\xi)} + \mathrm{i}\lambda G(x,\xi) \right] \gamma(\xi) ds, \quad x \in \Omega, \tag{4.15}$$

with $\lambda \neq 0$ a real number.

The exterior-interior connection. Let Ω the domain of the obstacle, and u_0 be the incident wave. Let v_j and $u_j = u_0 + v_j, j = 1, 2, 3$ be the scattered and total waves for (i) sound-soft; (ii) sound-hard; (iii) impedance problems:

$$\Delta v_j(x) + \kappa^2 v_j(x) = 0, \quad x \in \mathbb{R}^3 \setminus \overline{\Omega}, \tag{4.16}$$

$$u_1 = 0,\ \partial_n u_2 = 0,\ (\partial_n + \mathrm{i}\lambda) u_3 = 0, \quad x \in \partial\Omega. \tag{4.17}$$

Together with the Sommerfeld radiation condition at the infinity, these boundary value problems are well-posed for real numbers $\kappa > 0, \lambda \neq 0$. It follows immediately from the Green's representation theorem

$$v(x) = -\int_{\partial\Omega} \left[\partial_n u(\xi) G(x,\xi) - u(\xi) \frac{\partial G(x,\xi)}{\partial n(\xi)} \right] ds(\xi)$$

and that the scattered waves can be expressed as the layer potentials of single, double, and combined types

$$v_1(x) = \int_{\partial\Omega} G(x,\xi)a(\xi)ds, \tag{4.18}$$

$$v_2(x) = \int_{\partial\Omega} \frac{\partial G(x,\xi)}{\partial n(\xi)} b(\xi)ds, \tag{4.19}$$

$$v_3(x) = \int_{\partial\Omega} \left[\frac{\partial G(x,\xi)}{\partial n(\xi)} + \mathrm{i}\lambda G(x,\xi)\right] c(\xi)ds, \tag{4.20}$$

and that the densities are

$$a = -\partial_n u_1,\ b = u_2,\ c = u_3. \tag{4.21}$$

Remark 4.3.2. *The densities a, b, c are referred to as the naturally induced sources for the sound-soft, sound-hard, and impedance obstacles, respectively.*

The three exterior obstacles scattering problems of determining the naturally induced sources a, b, c are identical (up to a sign) to the three interior problems of determining the monopole, dipole, and combined equivalent sources.

Theorem 4.3.3. *Let $\kappa > 0, \lambda \neq 0$ be real numbers, and $\partial\Omega$ be smooth. Let α, β, and γ be the equivalent sources of monopole, dipole, and combined types for the incident wave u_0. Let a, b, c be the naturally induced sources. Then*

$$a = -\alpha,\ b = -\beta,\ c = -\gamma. \tag{4.22}$$

Proof. These are direct consequences of the jump conditions: $v_1, \partial_n v_2$, and $(\partial_n + \mathrm{i}\lambda)v_3$ are all continuous across $\partial\Omega$. To illustrate, we only prove the last one, and it remains to verify the impedance boundary condition for v. Indeed,

$$\begin{aligned}(\partial_n + \mathrm{i}\lambda)v_3(x^+) &= -(\partial_n + \mathrm{i}\lambda)\int_{\partial\Omega}\left[\frac{\partial G(x^+,\xi)}{\partial n(\xi)} + \mathrm{i}\lambda G(x^+,\xi)\right]\gamma(\xi)ds \\ &= -(\partial_n + \mathrm{i}\lambda)\left[\frac{\partial G(x^-,\xi)}{\partial n(\xi)} + \mathrm{i}\lambda G(x^-,\xi)\right]\gamma(\xi)ds \\ &= -(\partial_n + \mathrm{i}\lambda)u_0(x^-) = -(\partial_n + \mathrm{i}\lambda)u_0(x),\end{aligned}$$

which completes the proof of this theorem. □

We will use the equivalent sources to rewrite boundary integral equations for obstacle scattering. The reformulated problems are convenient to solve, or their solution easier to process, than the standard approaches. The integral equations presented here are not structurally new, but they explore the flexibility in reinterpreting the incident wave as the boundary data to rearrange the solution process.

Therefore and again, we assume that the incident wave u_0 is generated by known sources away from the boundary, so that both u_0 and ∂u_0 can be evaluated on the boundary.

Sound-soft problem The problem is to determine density a which is equivalent to determine the monopole α. Applying $\partial_n + \mathrm{i}\mu$ to (4.13) from the interior side of Ω, we obtain the second kind integral equation for α:

$$\frac{\alpha(x)}{2} + \int_{\partial\Omega} \left[\frac{\partial G(x,\xi)}{\partial n(x)} + \mathrm{i}\mu G(x,\xi) \right] \alpha(\xi) ds = (\partial_n + \mathrm{i}\mu) u_0, \tag{4.23}$$

where μ is the combination coefficient of the single and double layer potentials. A nonzero real μ makes the above integral equation uniquely solvable.

Sound-hard and impedance problems In the following treatment, the sound-hard problem is a special case of the impedance problem with $\lambda = 0$; we will thus only consider the latter.

We restrict x on $\partial\Omega$ in (4.15) and obtain the second kind equation for γ:

$$-\frac{\gamma(x)}{2} + \int_{\partial\Omega} \left[\frac{\partial G(x,\xi)}{\partial n(\xi)} + \mathrm{i}\lambda G(x,\xi) \right] \gamma(\xi) ds = u_0(x). \tag{4.24}$$

Since the equation always has a solution $\gamma = u_3$, resonances occur if and only if κ is an interior Dirichlet eigenvalue.