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An $\boldsymbol{H}$-ψ formulation for the three-dimensional eddy current problem in laminated structures

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ABSTRACT

It is a very challenging problem for the direct simulation of the three-dimensional eddy currents in grain-oriented (GO) silicon steel laminations since the coating film is only several microns thick over each lamination and the magnetic permeability is nonlinear and anisotropic. In addition, the system of GO silicon steel laminations has multiple scales and the ratio of the largest scale to the smallest scale can be up to 10^6. In this paper, we study an $\boldsymbol{H}$-ψ formulation for the nonlinear eddy current problem in laminated conductors. By omitting the insulating films between neighboring laminations, we propose an approximate but effective $\boldsymbol{H}$-ψ formulation for the nonlinear eddy current problem, which reduces the scale ratio by 2–3 orders of magnitude. The well-posedness of the original problem and the approximate problem are established by examining their weak formulations. The convergence is proved for the solution of the approximate problem to the solution of the original problem as the thickness of coating films approaches zero.

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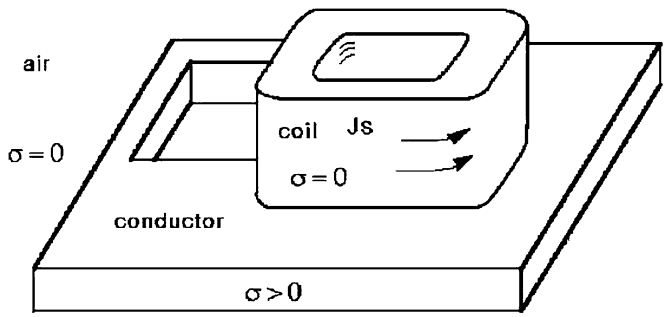

Fig. 1. A typical model of the eddy current problem.

1. Introduction

Consider the following eddy current problem for magnetic and anisotropic materials in terms of Farady's law and Ampere's law:

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\, \boldsymbol{E} = 0 \quad \text{in } \mathbb{R}^3, \tag{1a}$$

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} \quad \text{in } \mathbb{R}^3, \tag{1b}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic flux, $\boldsymbol{H}$ is the magnetic field, and $\boldsymbol{J}$ is the current density defined by

$$\boldsymbol{J} = \begin{cases} \sigma \boldsymbol{E} & \text{in } \Omega_c, \\ \boldsymbol{J}_s & \text{in } \mathbb{R}^3 \backslash \bar{\Omega}_c. \end{cases} \tag{2}$$

Eq. (1) is the eddy current model which approximates Maxwell's equations at very low frequency by neglecting the displacement currents in Ampere's law [1]. For magnetic materials, $\boldsymbol{B} = (B_1, B_2, B_3)$ is a nonlinear vector function of $\boldsymbol{H} = (H_1, H_2, H_3)$ in the form of $B_i = B_i(H_i)$, $1 \leqslant i \leqslant I$, so that (1) is the nonlinear eddy current problem. And for nonmagnetic materials, $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ where μ_0 is the permeability in the empty space, and (1) stands for the linear eddy current problem. In (2), $\sigma \geqslant 0$ is the electric conductivity, $\boldsymbol{J}_s$ is the source current density carried by some coils and satisfies div $\boldsymbol{J}_s = 0$, Ω_c denotes the conducting region, and the complement $\mathbb{R}^3 \backslash \bar{\Omega}_c$ denotes the nonconducting region (see Fig. 1). There are many works studying linear eddy current problems, e.g., on numerical methods [7,8,16,22,24,29], or on the regularity of the solution [17]. However, little has been done for the mathematical and numerical analysis for nonlinear eddy current problems. We refer to Bachinger et al. [6] for the numerical analysis of nonlinear multi-harmonic eddy current problems in isotropic materials.

In this paper, we shall study the nonlinear eddy current problem in GO silicon steel laminations. GO silicon steel laminations are widely used in iron cores and shielding structures of large power transformers [13,14]. The complex structure is made of many laminated steel sheets and each sheet is about 0.18–0.35 mm thick. Moreover, each steel sheet is coated with a thin layer of insulating film with thickness 2–5 μm to prevent the electric current from flowing into its neighboring sheets, as seen in Fig. 2. Usually the lamination stack has multiple scale sizes and the ratio of the largest scale to the smallest scale can be up to 10^6. Clearly, it is extremely difficult to do the full three-dimensional finite element simulation for the model problem (1) due to extensive unknowns from meshing the laminations and the coating films. Very few works have been done on the computation of three-dimensional eddy currents inside the laminations in the literature.

In recent years, there are considerable efforts which have been devoted to developing efficient numerical methods for nonlinear eddy current problems in steel laminations in the engineering community. Most of them were particularly made for effective reluctivities and conductivities of the lamination stack, e.g. [9,10,23,27]. The main idea is to replace physical parameters with equivalent

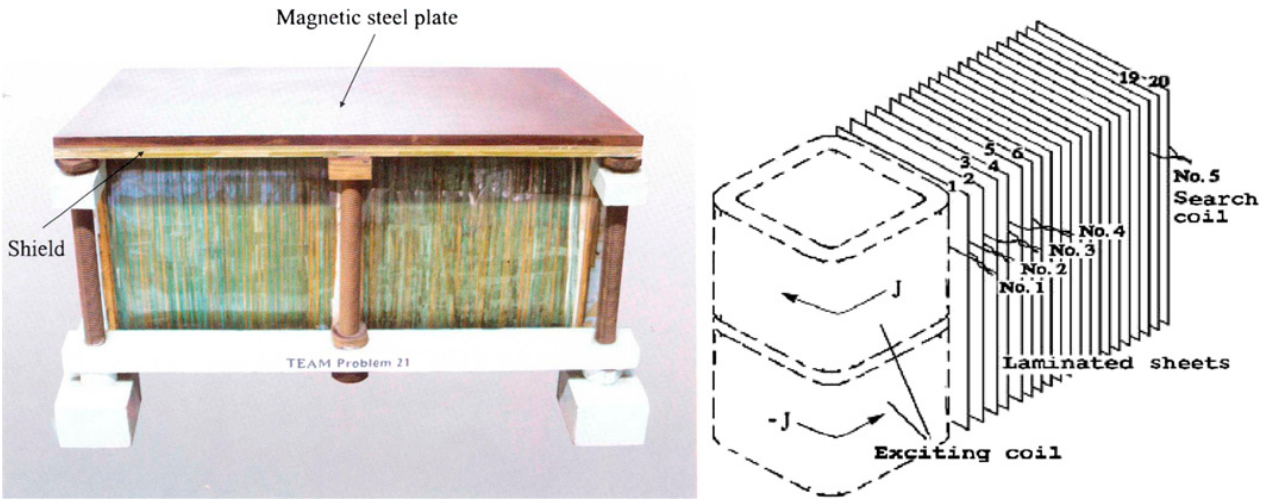

Fig. 2. GO silicon steel laminations. (Left) the magnetic shield for protecting the magnetic plate; (Right) the magnetic shield made of laminated steel sheets.

(or homogenized) parameters for Maxwell's equations. In [11,12], Bottauscio et al. proposed a mathematical homogenization technique based on the multi-scale expansion theory to derive the equivalent electric parameters and effective magnetization properties. In [19,20], Gyselinck et al. deduced the effective material parameters by an orthogonal decomposition of the flux in the perpendicular and parallel directions to the lamination plane. In [21], Napieralska-Juszczak et al. established equivalent characteristics of magnetic joints of transformer cores by minimizing the magnetic energy of the system. In [25], Nédélec and Wolf studied the homogenization method for eddy currents in a transformer core and proved the convergence of the exact solution to the solution of the homogenized problem as the thickness of the steel sheet approaches zero. Numerical methods based on the homogenization of material parameters provide an efficient way to simulate electromagnetic field in steel laminations. In [2], Ammari and Nédélec studied the electromagnetic scattering problem by a perfectly conducting object coated with a thin chiral curved layer. They proposed the approximate impedance condition without modeling the exact fields inside the thin layer and proved optimal error estimates. In [3], Ammari et al. studied the electromagnetic scattering by a thin dielectric planar structure. The approximate solution of Maxwell's equations comprises both the leading term and a first-order correction. We also refer to [4] for the asymptotic analysis of nonlinear Maxwell's equations with thin ferromagnetic films.

Since the effective conductivity is anisotropic and has zero value in the perpendicular direction to the lamination plane, the homogenized eddy current is thus two-dimensional in the lamination stack. Moreover, since the number of steel laminations is finite, the homogenization method usually introduces large modeling error near the boundary of the lamination stack, especially near the part of boundary close to the applied field. When the leakage of the magnetic flux is so strong as to enter the lamination plane perpendicularly, for example, in the outer laminations of large power transformer core, the eddy current loss induced there must be taken into account in the electromagnetic design. It is preferable to accurately compute the three-dimensional eddy currents at least in a few laminations close to the source, i.e., to use the zoned treatment for practical approaches, as seen in Fig. 3. In the three-dimensional eddy current region, one usually has to subdivide the laminations and the coating films into fine meshes. Using the zoned treatment, Cheng et al. investigated in [15] the effect of the eddy current, induced by the normal magnetic field on the total iron loss and the distortion of the local magnetic flux in the lamination stack.

The purpose of this work is to present an approximate and effective model to the eddy current problem (1) by omitting coating films in the system. The new model reduces the scale ratio of the system by 2–3 orders of magnitude and thus can save computational efforts greatly in numerical approximations. Besides, the new model conserves eddy current inside each lamination even ignoring the coating films. The eddy current cannot flow across the interface between neighboring steel

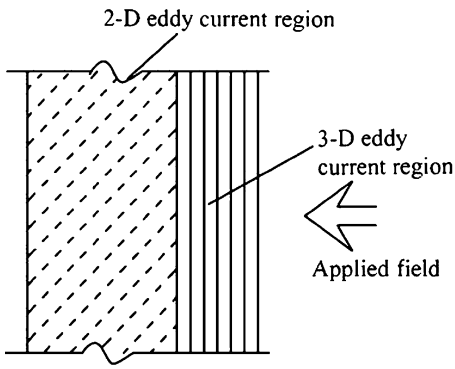

Fig. 3. Zoned treatment of the lamination stack. The six laminations close to the source are treated with three-dimensional simulations while the other laminations are treated with homogenization methods.

laminations. Specifically, we obtained the following results for Maxwell's equations of the nonlinear eddy current problem:

1. We proved the existence and uniqueness of both the exact solution and the approximate solution for the original and the approximate problems. We developed some new techniques to handle the nonlinearity in the mathematical analysis of Maxwell's equations. To the best of our knowledge, this is the first work studying the well-posedness of the $\boldsymbol{H}$-ψ formulation for nonlinear and time-dependent eddy current problems.
2. We proved the stability of the exact and the approximate solutions to the original and the approximate problems with respect to the source current.
3. We proved that the approximate solution converges strongly to the exact solution in the $\boldsymbol{L}^2$-norm as the thickness of the coating film approaches zero.
4. For the linear eddy current problem, we deduced an explicit error estimate between the approximate solution and the exact solution with respect to the thickness of the coating film.

The layout of the paper is as follows. In Section 2, we present some notation and Sobolev spaces and study the $\boldsymbol{H}$-ψ formulation for the model problem (1). The well-posedness of the nonlinear eddy current problem is established in Section 3. Section 4 is devoted to the well-posedness of the solution for an approximate $\boldsymbol{H}$-ψ formulation of the nonlinear eddy current problem by omitting coating films. The convergence is examined in Section 5 for the approximate solution to the exact solution as the thickness of the coating film tends to zero.

2. The $\boldsymbol{H}$-ψ formulation of eddy current problem

Let $\Omega \subset \mathbb{R}^3$ be a sufficiently large, bounded, and convex polyhedral domain containing all conductors and coils. Denote the conducting domain by Ω_c which consists of all conductors. Let $\Omega_{nc} = \Omega \backslash \overline{\Omega}_c$ be the nonconducting domain such that $\sigma \equiv 0$ in Ω_{nc}. Throughout the paper, we make the following assumptions on the electric conductivity and the nonlinear relationship between $\boldsymbol{H} = (H_1, H_2, H_3)$ and $\boldsymbol{B} = (B_1, B_2, B_3)$ which are usually satisfied in electrical engineering:

(H1) The conductivity σ is a piecewise constant in Ω. There exist two constants $\sigma_{\min}$ and $\sigma_{\max}$ such that

$$0 < \sigma_{\min} \leqslant \sigma \leqslant \sigma_{\max} \quad \text{in } \Omega_c.$$

(H2) B_i is a Lipschitz continuous function of H_i satisfying $B_i(H_i) = \mu_0 H_i$ in Ω_{nc} and $B_i(0) = 0$. There exist two constants $\mu_{\min}$ and $\mu_{\max}$ such that

$$0 < \mu_{\min} \leqslant B_i'(H_i) \leqslant \mu_{\max} \quad \text{a.e. in } \Omega,\ i = 1, 2, 3.$$

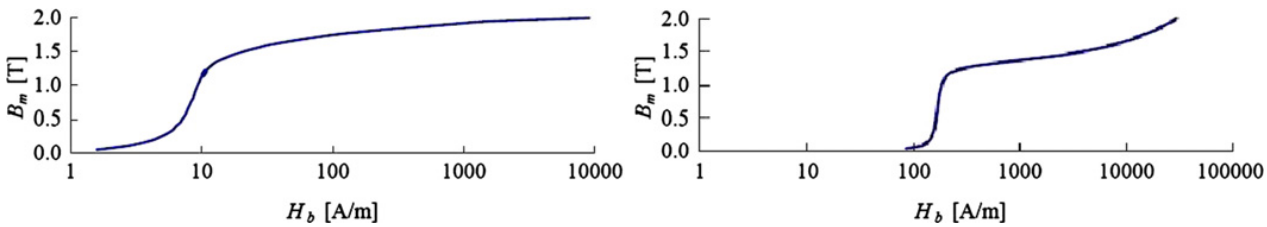

Fig. 4. BH-curves for GO silicon steel laminations. (Left) the rolling direction; (Right) the transverse direction.

Here μ_0 is the magnetic permeability of the vacuum. The nonlinear function $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{B})$ is usually obtained by spline interpolations using experimental data. Fig. 4 shows a typical example of the BH-curves for the GO silicon steel laminations in large power transformers [13]. Clearly, the assumption (H2) is satisfied.

We shall focus on simply-connected conductors, i.e., each connected component of Ω_c is a simply-connected Lipschitz domain. A typical engineering application lies in magnetic shields for the oil tank of large power transformers. We also refer to [13] for a family of benchmark problems, TEAM Workshop Problem P21^c-M1, P21^c-EM1, P21^c-M2, P21^c-EM2, and P21^d-M, for this application, as seen in Fig. 2. Obviously, the nonconducting domain Ω_{nc} is simply connected for these benchmark problems.

2.1. Hilbert spaces

Let $L^2(\Omega)$ be the usual Hilbert space of square integrable functions equipped with the following inner product and norm:

$$(u, v) := \int_\Omega u(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \quad \text{and} \quad \|u\|_{0,\Omega} := (u, u)^{1/2}.$$

Define $H^m(\Omega) := \{v \in L^2(\Omega)\colon\ D^\xi v \in L^2(\Omega),\ |\xi| \leqslant m\}$ which is equipped with the following norm and semi-norm

$$\|u\|_{m,\Omega} := \Big(\sum_{|\xi| \leqslant m} \|D^\xi u\|_{0,\Omega}^2 \Big)^{1/2} \quad \text{and} \quad |u|_{m,\Omega} := \Big(\sum_{|\xi| = m} \|D^\xi u\|_{0,\Omega}^2 \Big)^{1/2},$$

where ξ represents non-negative triple index. As usual, $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ whose functions have zero traces on $\partial\Omega$. Throughout, we denote vector-valued quantities by boldface notation, such as $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^3$.

Define

$$\boldsymbol{H}(\mathbf{curl}, \Omega) := \big\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\colon\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\big\},$$
$$\boldsymbol{H}_0(\mathbf{curl}, \Omega) := \big\{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)\colon\ \boldsymbol{n} \times \boldsymbol{v} = 0 \text{ on } \partial\Omega\big\},$$

where $\boldsymbol{n}$ is the unit outer normal and the spaces are equipped with the inner product

$$(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{H}(\mathbf{curl},\Omega)} := (\boldsymbol{v}, \boldsymbol{w}) + (\mathbf{curl}\, \boldsymbol{v}, \mathbf{curl}\, \boldsymbol{w})$$

and the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} := \sqrt{(\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{H}(\mathbf{curl},\Omega)}}.$$

Introduce the spaces of functions with square integrable divergence

$$\boldsymbol{H}(\mathrm{div}, \Omega) := \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\colon \mathrm{div}\, \boldsymbol{v} \in L^2(\Omega)\right\},$$
$$\boldsymbol{H}_0(\mathrm{div}, \Omega) := \left\{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)\colon \boldsymbol{n} \cdot \boldsymbol{v} = 0 \text{ on } \partial\Omega\right\},$$

which are equipped with the inner product

$$(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{H}(\mathrm{div},\Omega)} := (\boldsymbol{v}, \boldsymbol{w}) + (\mathrm{div}\, \boldsymbol{v}, \mathrm{div}\, \boldsymbol{w})$$

and the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} := \sqrt{(\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{H}(\mathrm{div},\Omega)}}.$$

To study the weak solution of (1), we shall use the subspaces of $\boldsymbol{H}(\mathbf{curl}, \Omega)$:

$$\boldsymbol{X} := \nabla H^1(\Omega) + \boldsymbol{H}_0(\mathbf{curl}, \Omega_c), \tag{3}$$
$$\boldsymbol{X}_c := \left\{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_c)\colon \mathrm{div}\, \boldsymbol{v} = 0 \text{ in } \Omega_c\right\}. \tag{4}$$

It is easy to see that

$$\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)} = \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega_c)} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}.$$

We shall use the convention that all functions in $\boldsymbol{H}_0(\mathbf{curl}, D)$ and $H^1_0(D)$ are extended by zero to the exterior of D for any $D \subset \Omega$.

Lemma 2.1. *Let the nonconducting region Ω_{nc} be simply connected. Then*

$$\boldsymbol{X} = \left\{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)\colon \mathbf{curl}\, \boldsymbol{v} = 0 \textit{ in } \Omega_{nc}\right\}.$$

Furthermore, for any $\boldsymbol{v} \in \boldsymbol{X}$, there exist a unique $\boldsymbol{v}_c \in \boldsymbol{X}_c$ and a unique $\phi \in H^1(\Omega)/\mathbb{R}$ such that

$$\boldsymbol{v} = \boldsymbol{v}_c + \nabla\phi, \qquad \|\boldsymbol{v}_c\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} + \|\phi\|_{H^1(\Omega)} \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)},$$

where $C > 0$ is a constant only depending on Ω_{nc}.

Proof. It is clear that $\boldsymbol{X} \subset \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)\colon \mathbf{curl}\, \boldsymbol{v} = 0 \text{ in } \Omega_{nc}\}$.

Suppose that $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ and satisfies $\mathbf{curl}\, \boldsymbol{v} = 0$ in Ω_{nc}. Since Ω_{nc} is simply connected, the potential theorem [5] shows that $\boldsymbol{v} = \nabla\phi_{nc}$ in Ω_{nc} for some $\phi_{nc} \in H^1(\Omega_{nc})$. By Stein's extension theorem [28], there exist a $\phi_1 \in H^1(\Omega)$ and a constant C only depending on Ω_{nc} such that

$$\phi_1 = \phi_{nc} \quad \text{in } \Omega_{nc} \quad \text{and} \quad \|\phi_1\|_{H^1(\Omega)} \leqslant C\|\phi_{nc}\|_{H^1(\Omega_{nc})} \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}. \tag{5}$$

Clearly $\boldsymbol{v}_1 := \boldsymbol{v} - \nabla\phi_1 \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_c)$ and satisfies

$$\|\boldsymbol{v}_1\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leqslant C\big(\|\phi_1\|_{H^1(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\big) \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}. \tag{6}$$

Thus $\boldsymbol{v} = \boldsymbol{v}_1 + \nabla\phi_1 \in \boldsymbol{X}$ and $\{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)\colon \mathbf{curl}\, \boldsymbol{v} = 0 \text{ in } \Omega_{nc}\} \subset \boldsymbol{X}$.

Now suppose that $\boldsymbol{v} = \boldsymbol{v}_1 + \nabla\phi_1$ with $\boldsymbol{v}_1 \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_c)$ and $\phi_1 \in H^1(\Omega)/\mathbb{R}$. Let $\phi_2 \in H_0^1(\Omega_c)$ be the unique solution of the following elliptic problem:

$$\int_{\Omega_c} \nabla\phi_2 \cdot \nabla\varphi = \int_{\Omega_c} \boldsymbol{v}_1 \cdot \nabla\varphi \quad \text{for all } \varphi \in H_0^1(\Omega_c).$$

Then $\boldsymbol{v} = \boldsymbol{v}_c + \nabla\phi$ with $\boldsymbol{v}_c := \boldsymbol{v}_1 - \nabla\phi_2 \in \boldsymbol{X}_c$ and $\phi := \phi_1 + \phi_2 \in H^1(\Omega)/\mathbb{R}$. Combining (5) and (6) yields that

$$\|\boldsymbol{v}_c\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leqslant 2\|\boldsymbol{v}_1\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}$$

and

$$\|\phi\|_{H^1(\Omega)} \leqslant C\|\nabla\phi\|_{\boldsymbol{L}^2(\Omega)} \leqslant C\|\boldsymbol{v} - \boldsymbol{v}_c\|_{L^2(\Omega)} \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}.$$

To prove the uniqueness, we let $\boldsymbol{v} = \tilde{\boldsymbol{v}}_c + \nabla\tilde{\phi}$ be another decomposition with $\tilde{\boldsymbol{v}}_c \in \boldsymbol{X}_c$ and $\tilde{\phi} \in H^1(\Omega)/\mathbb{R}$. Then $\boldsymbol{v}_c - \tilde{\boldsymbol{v}}_c \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_c)$ and satisfies

$$\operatorname{div}(\boldsymbol{v}_c - \tilde{\boldsymbol{v}}_c) = 0, \qquad \mathbf{curl}(\boldsymbol{v}_c - \tilde{\boldsymbol{v}}_c) = 0 \quad \text{in } \Omega_c.$$

It follows from [5] that $\boldsymbol{v}_c = \tilde{\boldsymbol{v}}_c$ and thus $\phi = \tilde{\phi}$. □

2.2. The weak formulation

Since $\operatorname{div} \boldsymbol{J}_s \equiv 0$, there exists a source magnetic field $\boldsymbol{H}_s$ such that

$$\boldsymbol{J}_s = \mathbf{curl}\, \boldsymbol{H}_s \quad \text{in } \mathbb{R}^3. \tag{7}$$

The field $\boldsymbol{H}_s$ can be written explicitly by the Biot–Savart law for general coils

$$\boldsymbol{H}_s := \mathbf{curl}\, \boldsymbol{A}_s,$$

where

$$\boldsymbol{A}_s(\boldsymbol{x}) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\boldsymbol{J}_s(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|}\, \mathrm{d}\boldsymbol{y}.$$

Denote the residual $\boldsymbol{H}_r := \boldsymbol{H} - \boldsymbol{H}_s$, which is also called the reaction field in [18]. Using (1) and (7), we have $\mathbf{curl}\, \boldsymbol{H}_r = 0$ in Ω_{nc}. A direct application of Lemma 2.1 yields the following result.

Lemma 2.2. *The reaction field $\boldsymbol{H}_r \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ admits a unique decomposition*

$$\boldsymbol{H}_r = \boldsymbol{u} + \nabla\psi, \quad \boldsymbol{u} \in \boldsymbol{X}_c,\ \psi \in H^1(\Omega)/\mathbb{R}. \tag{8}$$

Next we deduce a weak formulation of (1). For any $\boldsymbol{v} \in \boldsymbol{X}$, (1a) implies that

$$\int_{\Omega_c} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{v} = -\int_{\Omega_c} \mathbf{curl}\, \boldsymbol{E} \cdot \boldsymbol{v} = -\int_{\Omega_c} \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{v} + \int_{\partial\Omega_c} (\boldsymbol{E} \times \boldsymbol{n}_c) \cdot \boldsymbol{v}$$

$$= \int_{\Omega_c} \sigma^{-1}(\boldsymbol{J}_s - \mathbf{curl}\, \boldsymbol{H}) \cdot \mathbf{curl}\, \boldsymbol{v} + \int_{\partial\Omega_c} (\boldsymbol{E} \times \boldsymbol{n}_c) \cdot \boldsymbol{v}, \tag{9}$$

$$\int_{\Omega_{nc}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{v} = -\int_{\Omega_{nc}} \mathbf{curl}\, \boldsymbol{E} \cdot \boldsymbol{v} = \int_{\partial\Omega_{nc}} (\boldsymbol{E} \times \boldsymbol{n}_{nc}) \cdot \boldsymbol{v}, \tag{10}$$

where $\boldsymbol{n}_c$, $\boldsymbol{n}_{nc}$ are the unit outer normals to $\partial\Omega_c$ and $\partial\Omega_{nc}$ respectively. Noting the tangential continuity of $\boldsymbol{E}$ and $\boldsymbol{v}$ across $\partial\Omega_c$ and the fact that

$$\sigma^{-1}(\boldsymbol{J}_s - \mathbf{curl}\, \boldsymbol{H}) = -\sigma^{-1}\, \mathbf{curl}\, \boldsymbol{H}_r \quad \text{in } \Omega_c,$$

we add (9) and (10) and obtain

$$\int_{\Omega} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{v} + \int_{\Omega_c} \sigma^{-1}\, \mathbf{curl}\, \boldsymbol{H}_r \cdot \mathbf{curl}\, \boldsymbol{v} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}. \tag{11}$$

For convenience, we shall drop the subscript of $\boldsymbol{H}_r$ and let $\boldsymbol{H}$ denote the reaction field in the rest of the paper, and define

$$\sigma_1 = \begin{cases} \sigma^{-1} & \text{in } \Omega_c, \\ 0 & \text{in } \Omega_{nc}. \end{cases}$$

Using (11) and viewing $\boldsymbol{B}$ as a nonlinear vector function of the total magnetic field, we obtain a weak formulation of (1): Find $\boldsymbol{H} \in \boldsymbol{X}$ such that $\boldsymbol{H}(\cdot, 0) = 0$ and

$$\int_{\Omega} \frac{\partial}{\partial t} \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}_s) \cdot \boldsymbol{v} + \int_{\Omega} \sigma_1\, \mathbf{curl}\, \boldsymbol{H} \cdot \mathbf{curl}\, \boldsymbol{v} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}. \tag{12}$$

3. Well-posedness of the weak formulation

We shall use Rothe's method (cf. [26]) to study the weak solution. Let N be a positive integer and $\{t_n = n\tau\colon\ n = 0, \ldots, N\}$ be an equidistant partition of $[0, T]$ with $\tau = T/N$. The semi-discrete approximation to (12) reads: Given $\boldsymbol{H}_0 = 0$, find $\boldsymbol{H}_n \in \boldsymbol{X}$, $1 \leqslant n \leqslant N$, such that

$$\int_{\Omega} \frac{\boldsymbol{B}_n - \boldsymbol{B}_{n-1}}{\tau} \cdot \boldsymbol{v} + \int_{\Omega} \sigma_1\, \mathbf{curl}\, \boldsymbol{H}_n \cdot \mathbf{curl}\, \boldsymbol{v} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}, \tag{13}$$

where $\boldsymbol{B}_n := \boldsymbol{B}(\boldsymbol{H}_n + \boldsymbol{H}_s(t_n))$.

We define the piecewise constant and piecewise linear interpolations in time by

$$\bar{\boldsymbol{H}}_\tau(\cdot, t) = \boldsymbol{H}_n, \qquad \boldsymbol{H}_\tau(\cdot, t) = l_n(t)\boldsymbol{H}_n + \big(1 - l_n(t)\big)\boldsymbol{H}_{n-1},$$

$$\bar{\boldsymbol{B}}_\tau(\cdot, t) = \boldsymbol{B}_n, \qquad \boldsymbol{B}_\tau(\cdot, t) = l_n(t)\boldsymbol{B}_n + \big(1 - l_n(t)\big)\boldsymbol{B}_{n-1}, \tag{14}$$

for any $t \in (t_{n-1}, t_n]$ and $1 \leqslant n \leqslant N$, where

$$l_n(t) := \tau^{-1}(t - t_{n-1}).$$

Clearly we have

$$\bar{\boldsymbol{H}}_\tau \in \boldsymbol{L}^2(0, T; \boldsymbol{X}), \qquad \boldsymbol{H}_\tau \in \boldsymbol{C}(0, T; \boldsymbol{X}),$$
$$\bar{\boldsymbol{B}}_\tau \in \boldsymbol{L}^2\big(0, T; \boldsymbol{L}^2(\Omega)\big), \qquad \boldsymbol{B}_\tau \in \boldsymbol{C}\big(0, T; \boldsymbol{L}^2(\Omega)\big).$$

The following lemma is concerned with the well-posedness of the weak formulation (13). The proof is given in Appendix A.

Lemma 3.1. *For any $1 \leqslant n \leqslant N$, the weak formulation (13) has a unique solution $\boldsymbol{H}_n \in \boldsymbol{X}$. Suppose that $\boldsymbol{H}_s \in \boldsymbol{H}^1(0, T; \boldsymbol{L}^2(\Omega))$. Then there exists a constant C only depending on Ω_c, T, $\mu_{\max}$, $\mu_{\min}$, $\sigma_{\max}$, $\sigma_{\min}$ such that*

$$\|\boldsymbol{H}_\tau\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))} + \|\mathbf{curl}\, \boldsymbol{H}_\tau\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))} \leqslant C\|\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}, \tag{15}$$

$$\|\bar{\boldsymbol{H}}_\tau\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\mathbf{curl}\, \bar{\boldsymbol{H}}_\tau\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))} \leqslant C\|\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}, \tag{16}$$

$$\|\bar{\boldsymbol{B}}_\tau\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{B}_\tau\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))} \leqslant C\|\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}. \tag{17}$$

It follows from Lemma 2.1 that each $\boldsymbol{H}_n$ admits the decomposition in a direct sum

$$\boldsymbol{H}_n = \boldsymbol{u}_n + \nabla\psi_n, \quad \boldsymbol{u}_n \in \boldsymbol{X}_c,\ \psi_n \in H^1(\Omega)/\mathbb{R}.$$

For any $t \in (t_{n-1}, t_n]$ and $1 \leqslant n \leqslant N$, we may define $\bar{\boldsymbol{u}}_\tau \in \boldsymbol{L}^2(0, T; \boldsymbol{X}_c)$ and $\boldsymbol{u}_\tau \in \boldsymbol{C}(0, T; \boldsymbol{X}_c)$ by using $\{\boldsymbol{u}_n\}$:

$$\bar{\boldsymbol{u}}_\tau(\cdot, t) = \boldsymbol{u}_n, \qquad \boldsymbol{u}_\tau(\cdot, t) = l_n(t)\boldsymbol{u}_n + \big(1 - l_n(t)\big)\boldsymbol{u}_{n-1}, \tag{18}$$

which gives the decompositions

$$\bar{\boldsymbol{H}}_\tau = \bar{\boldsymbol{u}}_\tau + \nabla\bar{\psi}_\tau, \qquad \boldsymbol{H}_\tau = \boldsymbol{u}_\tau + \nabla\psi_\tau.$$

Here $\bar{\psi}_\tau$ and ψ_τ are defined as

$$\bar{\psi}_\tau(\cdot, t) := \psi_n, \qquad \psi_\tau(\cdot, t) := l_n(t)\psi_n + \big(1 - l_n(t)\big)\psi_{n-1},$$

for all $t \in (t_{n-1}, t_n]$ and $1 \leqslant n \leqslant N$.

Introduce the Sobolev–Bochner space (cf. [26, Section 7.1])

$$\boldsymbol{W}^{1,2,2}\big(0, T; \boldsymbol{H}^1(\Omega), \boldsymbol{L}^2(\Omega)\big) := \left\{\boldsymbol{v} \in \boldsymbol{L}^2\big(0, T; \boldsymbol{H}^1(\Omega)\big):\ \frac{\partial \boldsymbol{v}}{\partial t} \in \boldsymbol{L}^2\big(0, T; \boldsymbol{L}^2(\Omega)\big)\right\},$$

which is equipped with the following norm

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,2,2}(0,T;\boldsymbol{H}^1(\Omega),\boldsymbol{L}^2(\Omega))} = \left(\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{H}^1(\Omega))} + \left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}\right)^{1/2}.$$

Next we examine the convergence of the temporally discrete functions. For convenience, we shall use the same notation to denote their subsequences without causing confusion.

Lemma 3.2. *Assume that each connected component of Ω_c is a convex polyhedron. Then $\boldsymbol{u}_\tau \in \boldsymbol{W}^{1,2,2}(0,T;\boldsymbol{H}^1(\Omega_c),\boldsymbol{L}^2(\Omega_c))$ and satisfies*

$$\|\bar{\boldsymbol{u}}_\tau\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}^1(\Omega_c))} + \|\boldsymbol{u}_\tau\|_{\boldsymbol{W}^{1,2,2}(0,T;\boldsymbol{H}^1(\Omega_c),\boldsymbol{L}^2(\Omega_c))} \leqslant C\|\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}. \tag{19}$$

Furthermore, there exist a subsequence of $\{\boldsymbol{u}_\tau\}_{\tau\geqslant 0}$, a subsequence of $\{\bar{\boldsymbol{u}}_\tau\}_{\tau\geqslant 0}$, and a $\boldsymbol{u} \in \boldsymbol{L}^2(0,T;\boldsymbol{X}_c)$ such that

$$\lim_{\tau\to 0}\|\boldsymbol{u}_\tau - \boldsymbol{u}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega_c))} = \lim_{\tau\to 0}\|\bar{\boldsymbol{u}}_\tau - \boldsymbol{u}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega_c))} = 0. \tag{20}$$

Proof. Since each connected component of Ω_c is a convex polyhedron, we know that $\boldsymbol{H}_0(\mathbf{curl},\Omega_c)\cap \boldsymbol{H}(\mathrm{div},\Omega_c) \subset \boldsymbol{H}^1(\Omega_c)$ (cf. e.g., [5]). This implies $\boldsymbol{u}_n \in \boldsymbol{H}^1(\Omega_c)$ and

$$\|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\Omega_c)} \leqslant C\big\{\|\boldsymbol{u}_n\|_{\boldsymbol{H}(\mathbf{curl},\Omega_c)} + \|\mathrm{div}\,\boldsymbol{u}_n\|_{\boldsymbol{L}^2(\Omega_c)}\big\} \leqslant C\|\boldsymbol{H}_n\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}, \tag{21}$$

where we have used Lemma 2.1 and the fact that $\mathrm{div}\,\boldsymbol{u}_n = 0$ in Ω_c in the last inequality. Then (19) follows from (15).

By the compact embedding (cf. [26, Lemma 7.7])

$$\boldsymbol{W}^{1,2,2}\big(0,T;\boldsymbol{H}^1(\Omega_c),\boldsymbol{L}^2(\Omega_c)\big) \Subset \boldsymbol{L}^2\big(0,T;\boldsymbol{L}^2(\Omega_c)\big),$$

there exist a subsequence of $\{\boldsymbol{u}_\tau\}_{\tau\geqslant 0}$ and a $\boldsymbol{u} \in \boldsymbol{W}^{1,2,2}(0,T;\boldsymbol{H}^1(\Omega_c),\boldsymbol{L}^2(\Omega_c))$ such that

$$\lim_{\tau\to 0}\boldsymbol{u}_\tau = \boldsymbol{u} \quad \begin{array}{l}\text{strongly in } \boldsymbol{L}^2\big(0,T;\boldsymbol{L}^2(\Omega_c)\big),\\ \text{weakly in } \boldsymbol{W}^{1,2,2}\big(0,T;\boldsymbol{H}^1(\Omega_c),\boldsymbol{L}^2(\Omega_c)\big).\end{array}$$

The weak convergence of $\{\boldsymbol{u}_\tau\}_{\tau\geqslant 0}$ indicates that, for any $\boldsymbol{v} \in \boldsymbol{L}^2(0,T;\boldsymbol{H}^1(\Omega_c))$,

$$\begin{aligned}\int_0^T\int_{\partial\Omega_c}(\boldsymbol{u}\times\boldsymbol{n})\cdot\boldsymbol{v} &= \int_0^T\int_{\Omega_c}(\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} - \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{v}) = \lim_{\tau\to 0}\int_0^T\int_{\Omega_c}(\boldsymbol{u}_\tau\cdot\mathbf{curl}\,\boldsymbol{v} - \mathbf{curl}\,\boldsymbol{u}_\tau\cdot\boldsymbol{v})\\ &= \lim_{\tau\to 0}\int_0^T\int_{\partial\Omega_c}(\boldsymbol{u}_\tau\times\boldsymbol{n})\cdot\boldsymbol{v} = 0.\end{aligned}$$

We conclude that $\boldsymbol{u} \in \boldsymbol{L}^2(0,T;\boldsymbol{H}_0(\mathbf{curl},\Omega_c))$. Since $\mathrm{div}\,\boldsymbol{u}_\tau = 0$, it is clear that

$$\int_0^T\int_{\Omega_c}\boldsymbol{u}\cdot\nabla\varphi = \lim_{\tau\to 0}\int_0^T\int_{\Omega_c}\boldsymbol{u}_\tau\cdot\nabla\varphi = 0 \quad \text{for all } \varphi \in L^2\big(0,T;H_0^1(\Omega_c)\big),$$

which shows $\mathrm{div}\,\boldsymbol{u} = 0$ and thus $\boldsymbol{u} \in \boldsymbol{L}^2(0,T;\boldsymbol{X}_c)$.

Finally we deduce from (19) and the strong convergence of $\boldsymbol{u}_\tau$ that

$$\lim_{\tau\to 0}\|\bar{\boldsymbol{u}}_\tau-\boldsymbol{u}\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega_c))}\leqslant\lim_{\tau\to 0}\|\bar{\boldsymbol{u}}_\tau-\boldsymbol{u}_\tau\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega_c))}=\lim_{\tau\to 0}\sum_{n=1}^{N}\frac{\tau}{3}\|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}\|^2_{\boldsymbol{L}^2(\Omega_c)}$$
$$=\lim_{\tau\to 0}\frac{\tau^2}{3}\left\|\frac{\partial\boldsymbol{u}_\tau}{\partial t}\right\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega_c))}=0,$$

which completes the proof. □

Lemma 3.3. *Let* (H1)–(H2) *be satisfied. Assume that each connected component of Ω_c is a convex polyhedron and*

$$\lim_{\tau\to 0}\|\boldsymbol{H}_{s,\tau}-\boldsymbol{H}_s\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}=0, \tag{22}$$

where $\boldsymbol{H}_{s,\tau}$ is the piecewise constant interpolation of $\boldsymbol{H}_s$ in time

$$\boldsymbol{H}_{s,\tau}(\cdot,t)=\boldsymbol{H}_s(\cdot,t_n)\quad \textit{for all } t\in(t_{n-1},t_n],\ 1\leqslant n\leqslant N.$$

Then there exists an $\boldsymbol{H}\in\boldsymbol{L}^2(0,T;\boldsymbol{X})$ such that

$$\lim_{\tau\to 0}\boldsymbol{H}_\tau=\lim_{\tau\to 0}\bar{\boldsymbol{H}}_\tau=\boldsymbol{H}\quad\begin{array}{l}\textit{strongly in } \boldsymbol{L}^2\big(0,T;\boldsymbol{L}^2(\Omega)\big),\\ \textit{weakly in } \boldsymbol{L}^2(0,T;\boldsymbol{X}).\end{array}$$

Proof. Since $\boldsymbol{L}^2(0,T;\boldsymbol{X})$ is self-reflective, by (16), there exist a subsequence of $\{\boldsymbol{H}_\tau\}_{\tau\geqslant 0}$ and a subsequence of $\{\bar{\boldsymbol{H}}_\tau\}_{\tau\geqslant 0}$ such that

$$\lim_{\tau\to 0}\boldsymbol{H}_\tau=\lim_{\tau\to 0}\bar{\boldsymbol{H}}_\tau=\boldsymbol{H}\quad\text{weakly in } \boldsymbol{L}^2(0,T;\boldsymbol{X}).$$

By Lemma 2.1, $\boldsymbol{H}$ can be decomposed into $\boldsymbol{H}=\boldsymbol{u}+\nabla\psi$ where $\psi\in H^1(\Omega)/\mathbb{R}$ and $\boldsymbol{u}$ is the limit of $\bar{\boldsymbol{u}}_\tau$.

Next we prove the strong convergence of $\bar{\boldsymbol{H}}_\tau$. The strong convergence of $\boldsymbol{H}_\tau$ comes directly from that of $\bar{\boldsymbol{H}}_\tau$. For convenience we denote the discrete and continuous total magnetic fields by $\hat{\boldsymbol{H}}_\tau=\bar{\boldsymbol{H}}_\tau+\boldsymbol{H}_{s,\tau}$ and $\hat{\boldsymbol{H}}=\boldsymbol{H}+\boldsymbol{H}_s$ respectively. It follows from (22) and the weak convergence of $\bar{\boldsymbol{H}}_\tau$ that

$$\lim_{\tau\to 0}\hat{\boldsymbol{H}}_\tau=\hat{\boldsymbol{H}}\quad\text{weakly in } \boldsymbol{L}^2\big(0,T,\boldsymbol{L}^2(\Omega)\big). \tag{23}$$

From (13) and (14) we deduce that

$$(\bar{\boldsymbol{B}}_\tau,\nabla\varphi)=0\quad\text{for all } \varphi\in H^1(\Omega).$$

Then using Lemmas 3.1–3.2 and (22), we obtain

$$\lim_{\tau\to 0}\int_0^T(\bar{\boldsymbol{B}}_\tau,\hat{\boldsymbol{H}}_\tau-\hat{\boldsymbol{H}})=\lim_{\tau\to 0}\int_0^T(\bar{\boldsymbol{B}}_\tau,\bar{\boldsymbol{u}}_\tau-\boldsymbol{u}+\boldsymbol{H}_{s,\tau}-\boldsymbol{H}_s)=0. \tag{24}$$

Noting the monotonicity of $\boldsymbol{B}(\cdot)$ and using (23)–(24), we have

$$\mu_{\min} \lim_{\tau\to 0} \|\hat{\boldsymbol{H}}_\tau - \hat{\boldsymbol{H}}\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \leqslant \lim_{\tau\to 0} \int_0^T \big(\boldsymbol{B}(\hat{\boldsymbol{H}}_\tau) - \boldsymbol{B}(\hat{\boldsymbol{H}}), \hat{\boldsymbol{H}}_\tau - \hat{\boldsymbol{H}}\big)$$
$$= \lim_{\tau\to 0} \int_0^T (\bar{\boldsymbol{B}}_\tau, \hat{\boldsymbol{H}}_\tau - \hat{\boldsymbol{H}}) - \lim_{\tau\to 0} \int_0^T \big(\boldsymbol{B}(\hat{\boldsymbol{H}}), \hat{\boldsymbol{H}}_\tau - \hat{\boldsymbol{H}}\big) = 0,$$

which shows together with (22) that $\lim_{\tau\to 0} \|\bar{\boldsymbol{H}}_\tau - \boldsymbol{H}\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} = 0$. □

Theorem 3.1. *Let* (H1)–(H2) *be satisfied. Furthermore, assume that each connected component of Ω_c is a convex polyhedron, and $\boldsymbol{H}_s \in \boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))$ satisfies* (22) *and $\boldsymbol{H}_s|_{t=0} = 0$. Then* (12) *has a unique solution $\boldsymbol{H} \in \boldsymbol{H}^1(0,T;\boldsymbol{X})$, and there exists a constant C only depending on Ω_c, T, $\mu_{\max}$, $\mu_{\min}$, $\sigma_{\max}$, $\sigma_{\min}$ such that*

$$\|\boldsymbol{H}\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{H}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))} \leqslant C\|\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}.$$

Proof. Using (14), we first write (13) into the following equation

$$\int_0^T \left(\frac{\partial \boldsymbol{B}_\tau}{\partial t}, \boldsymbol{v}\right) + \int_0^T (\sigma_1 \,\mathbf{curl}\, \bar{\boldsymbol{H}}_\tau, \mathbf{curl}\, \boldsymbol{v}) = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2(0,T;\boldsymbol{X}). \tag{25}$$

Since $\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))$ is self-reflective, by (17), $\{\boldsymbol{B}_\tau\}_{\tau\geqslant 0}$ has a subsequence satisfying

$$\lim_{\tau\to 0} \boldsymbol{B}_\tau = \boldsymbol{B}_0 \quad \text{weakly in } \boldsymbol{H}^1\big(0,T;\boldsymbol{L}^2(\Omega)\big), \tag{26}$$

which implies that

$$\lim_{\tau\to 0} \bar{\boldsymbol{B}}_\tau = \boldsymbol{B}_0 \quad \text{weakly in } \boldsymbol{L}^2\big(0,T;\boldsymbol{L}^2(\Omega)\big).$$

Write $\hat{\boldsymbol{H}}_\tau := \bar{\boldsymbol{H}}_\tau + \boldsymbol{H}_{s,\tau}$ and $\hat{\boldsymbol{H}} := \boldsymbol{H} + \boldsymbol{H}_s$, where $\boldsymbol{H}$ is the limit of $\bar{\boldsymbol{H}}_\tau$ in Lemma 3.3. Due to the strong convergence of $\bar{\boldsymbol{H}}_\tau$ and $\boldsymbol{H}_{s,\tau}$, it is easy to show that

$$\lim_{\tau\to 0} \|\hat{\boldsymbol{H}}_\tau - \hat{\boldsymbol{H}}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} = 0.$$

Using (H2), we deduce that

$$\lim_{\tau\to 0} \big\|\bar{\boldsymbol{B}}_\tau - \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}_s)\big\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} = \lim_{\tau\to 0} \big\|\boldsymbol{B}(\hat{\boldsymbol{H}}_\tau) - \boldsymbol{B}(\hat{\boldsymbol{H}})\big\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}$$
$$\leqslant \mu_{\max} \lim_{\tau\to 0} \|\hat{\boldsymbol{H}}_\tau - \hat{\boldsymbol{H}}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} = 0.$$

Thus we have $\boldsymbol{B}_0 = \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}_s)$.

Since σ_1 is bounded and positive in Ω_c, the weak convergence of $\bar{\boldsymbol{H}}_\tau$ in $\boldsymbol{L}^2(0,T;\boldsymbol{X})$ shows that

$$\lim_{\tau\to 0} \int_0^T (\sigma_1 \,\mathbf{curl}\, \bar{\boldsymbol{H}}_\tau, \mathbf{curl}\, \boldsymbol{v}) = \int_0^T (\sigma_1 \,\mathbf{curl}\, \boldsymbol{H}, \mathbf{curl}\, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2(0,T;\boldsymbol{X}). \tag{27}$$

Plugging (26) and (27) into (25) lead to

$$\int_0^T \left(\frac{\partial}{\partial t} \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}_s), \boldsymbol{v} \right) + \int_0^T (\sigma_1 \, \mathbf{curl}\, \boldsymbol{H}, \mathbf{curl}\, \boldsymbol{v}) = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2(0, T; \boldsymbol{X}). \tag{28}$$

Therefore (12) holds in the sense of distribution.

Next we prove the initial condition. We write $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}_s)$ for convenience and take any $\varphi \in C^1([0, T])$ satisfying $\varphi(0) = 1$ and $\varphi(T) = 0$. By (26) and integration by parts, we deduce that

$$\begin{aligned}
(\boldsymbol{B}|_{t=0}, \boldsymbol{v}) &= \int_0^T \left\{ \varphi(t) \cdot \frac{\partial}{\partial t}(\boldsymbol{B}, \boldsymbol{v}) + \varphi'(t) \cdot (\boldsymbol{B}, \boldsymbol{v}) \right\} \mathrm{d}t \\
&= \lim_{\tau \to 0} \int_0^T \left\{ \varphi(t) \cdot \frac{\partial}{\partial t}(\boldsymbol{B}_\tau, \boldsymbol{v}) + \varphi'(t) \cdot (\boldsymbol{B}_\tau, \boldsymbol{v}) \right\} \mathrm{d}t \\
&= \lim_{\tau \to 0} (\boldsymbol{B}_\tau|_{t=0}, \boldsymbol{v}) = \big(\boldsymbol{B}(\boldsymbol{H}_0 + \boldsymbol{H}_s|_{t=0}), \boldsymbol{v} \big) = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).
\end{aligned}$$

Thus $\boldsymbol{B}|_{t=0} = 0$. Since $\boldsymbol{H}_s|_{t=0} = 0$, by (H2) we have $\boldsymbol{H}|_{t=0} = 0$.

The stability estimates for $\boldsymbol{H}$ are easy. In fact, from Lemma 3.1, there exists a subsequence of $\{\boldsymbol{H}_\tau\}_{\tau \geqslant 0}$ which converges to $\boldsymbol{H}$ weakly in both $\boldsymbol{H}^1(0, T; \boldsymbol{L}^2(\Omega))$ and $\boldsymbol{L}^2(0, T; \boldsymbol{X})$. Then (15) shows that

$$\begin{aligned}
&\left\| \frac{\partial \boldsymbol{H}}{\partial t} \right\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{H}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))} \\
&\quad \leqslant \varliminf_{\tau \to 0} \left\{ \left\| \frac{\partial \boldsymbol{H}_\tau}{\partial t} \right\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\boldsymbol{H}_\tau\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))} \right\} \\
&\quad \leqslant C \|\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))},
\end{aligned}$$

which completes the proof. □

4. The approximate formulation without coating films

GO silicon steel laminations are widely used in iron cores and magnetic shields of large power transformers. Each lamination is usually coated with an insulating film whose thickness is only 2–5 μm so that the electric current cannot flow into the neighboring laminations (see Fig. 5). In this section, we propose an approximate formulation by omitting coating films from the model. Comparing with traditional homogenization methods, the new model is an accurate approximation to the original problem and yields a full three-dimensional eddy current density inside laminations.

To simplify the setting, we assume that the conducting domain consists of hexahedral laminations, that is, $\Omega_c = \bigcup_{i=1}^{I} \Omega_i$ where

$$\begin{aligned}
\Omega_1 &:= (X_1, X_2) \times (Y_1, Y_2) \times (Z_0, Z_1), \\
\Omega_i &:= (X_1, X_2) \times (Y_1, Y_2) \times (Z_{i-1} + d, Z_i), \quad i = 2, 3, \ldots, I.
\end{aligned}$$

Here $d > 0$ stands for the thickness of the coating film (see Fig. 6). We assume that σ_1 is constant in each Ω_i, namely,

$$\sigma_1 \equiv C_i > 0 \quad \text{in } \Omega_i, \ 1 \leqslant i \leqslant I.$$

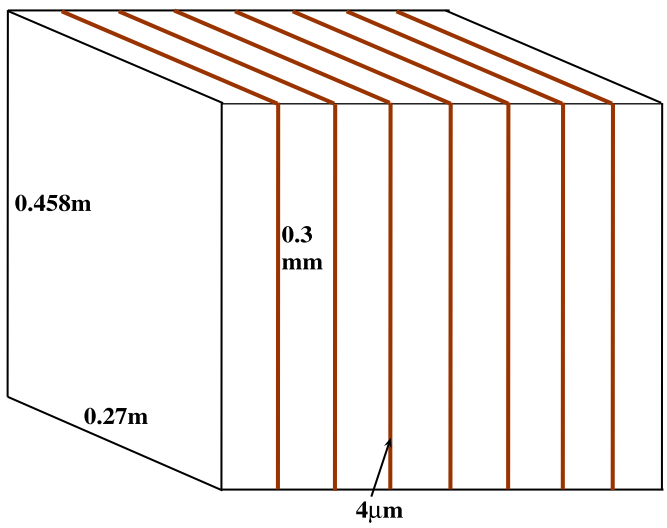

Fig. 5. Geometric size of silicon steel laminations in Team Workshop Problem 21^c-M1 [13].

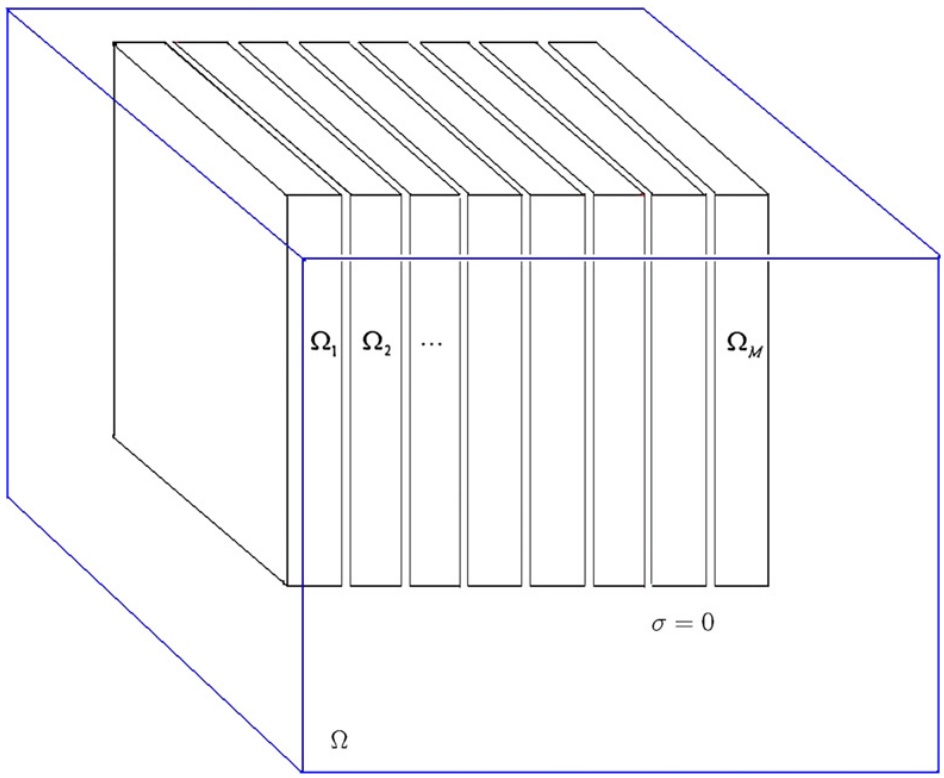

Fig. 6. Computational domain with steel laminations and coating films.

We remark that the assumptions on $\Omega_1, \ldots, \Omega_I$ and σ are not essential in the mathematical analysis. The results can be easily extended to convex polyhedral conductors and to the case that σ is not piecewise constant.

To omit coating films, we define the extended conductors by (see Fig. 7)

$$\tilde{\Omega}_c := (X_1, X_2) \times (Y_1, Y_2) \times (Z_0, Z_I),$$

$$\tilde{\Omega}_i := (X_1, X_2) \times (Y_1, Y_2) \times (Z_{i-1}, Z_i), \quad i = 1, 2, \ldots, I,$$

and define the modified material parameters by

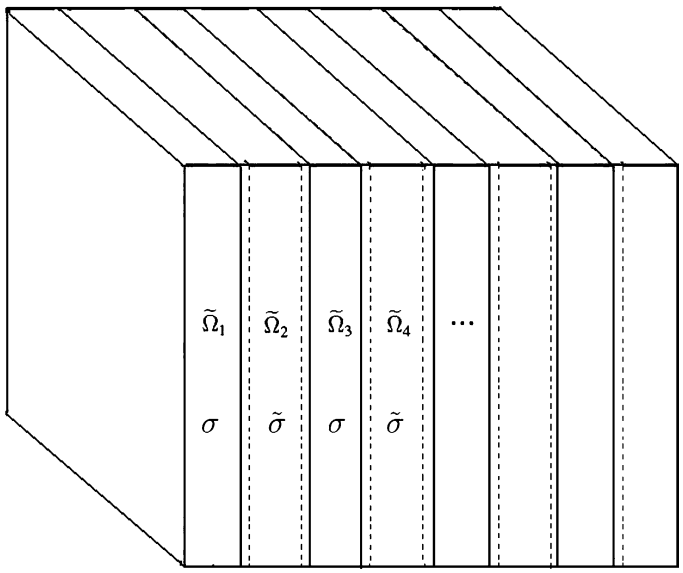

Fig. 7. Extended conductors by merging the coating film into the even conductors.

$$\tilde{\boldsymbol{B}}(\boldsymbol{H}) = \boldsymbol{B}(\boldsymbol{H}), \qquad \tilde{\sigma}_1 = C_i \quad \text{in } \tilde{\Omega}_i,\ 1 \leqslant i \leqslant I,$$
$$\tilde{\boldsymbol{B}}(\boldsymbol{H}) = \mu_0 \boldsymbol{H}, \qquad \tilde{\sigma}_1 = 0 \quad \text{elsewhere}. \tag{29}$$

The approximate formulation to (12) reads: Find $\tilde{\boldsymbol{H}} \in \tilde{\boldsymbol{X}}$ such that $\tilde{\boldsymbol{H}}(\cdot, 0) = 0$ and

$$\int_{\Omega} \frac{\partial}{\partial t} \tilde{\boldsymbol{B}}(\tilde{\boldsymbol{H}} + \boldsymbol{H}_s) \cdot \boldsymbol{v} + \int_{\Omega} \tilde{\sigma}_1 \,\mathbf{curl}\, \tilde{\boldsymbol{H}} \cdot \mathbf{curl}\, \boldsymbol{v} = 0 \quad \text{for all } \boldsymbol{v} \in \tilde{\boldsymbol{X}}, \tag{30}$$

where

$$\tilde{\boldsymbol{X}} := \nabla H^1(\Omega) + \sum_{i=1}^{I} \boldsymbol{H}_0(\mathbf{curl}, \tilde{\Omega}_i).$$

Here we adopt the convention that each function in $\boldsymbol{H}_0(\mathbf{curl}, \tilde{\Omega}_i)$ is extended by zero to the exterior of $\tilde{\Omega}_i$.

Lemma 4.1. *Define $\tilde{\boldsymbol{X}}_i := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \tilde{\Omega}_i)\colon \operatorname{div} \boldsymbol{v} = 0$ in $\tilde{\Omega}_i\}$ for any $1 \leqslant i \leqslant I$. Then any function $\boldsymbol{v} \in \tilde{\boldsymbol{X}}$ admits a unique decomposition*

$$\boldsymbol{v} = \nabla \psi + \sum_{i=1}^{I} \boldsymbol{v}_i, \quad \psi \in H^1(\Omega)/\mathbb{R},\ \boldsymbol{v}_i \in \tilde{\boldsymbol{X}}_i,$$
$$\|\nabla \psi\|_{\boldsymbol{L}^2(\Omega)} + \sum_{i=1}^{I} \|\boldsymbol{v}_i\|_{\boldsymbol{H}(\mathbf{curl}, \tilde{\Omega}_i)} \leqslant C \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega)},$$

where the constant $C > 0$ only depends on $\tilde{\Omega}_1, \ldots, \tilde{\Omega}_I$.

Proof. Let $\boldsymbol{v} = \sum_{i=1}^{I} \boldsymbol{w}_i + \nabla \phi \in \tilde{\boldsymbol{X}}$ be any function with $\boldsymbol{w}_i \in \boldsymbol{H}_0(\mathbf{curl}, \tilde{\Omega}_i)$ and $\phi \in H^1(\Omega)$. Let $\phi_i \in H_0^1(\tilde{\Omega}_i)$ solve the elliptic problems

$$\int_{\tilde{\Omega}_i} \nabla \phi_i \cdot \nabla \varphi = \int_{\tilde{\Omega}_i} \boldsymbol{w}_i \cdot \nabla \varphi \quad \text{for all } \varphi \in H_0^1(\tilde{\Omega}_i),\ 1 \leqslant i \leqslant I.$$

Then we have $\boldsymbol{v} = \sum_{i=1}^{I} \boldsymbol{v}_i + \nabla\psi$ with $\boldsymbol{v}_i := \boldsymbol{w}_i - \nabla\phi_i \in \tilde{\boldsymbol{X}}_i$ and $\psi = \phi + \sum_{i=1}^{I} \phi_i \in H^1(\Omega)$. The Poincaré-type inequality [5] shows that

$$\|\boldsymbol{v}_i\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leqslant C\|\mathbf{curl}\,\boldsymbol{v}_i\|_{\boldsymbol{L}^2(\tilde{\Omega}_i)} = C\|\mathbf{curl}\,\boldsymbol{w}_i\|_{\boldsymbol{L}^2(\tilde{\Omega}_i)} = C\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\tilde{\Omega}_i)},$$

$$\|\nabla\psi\|_{\boldsymbol{L}^2(\Omega)} \leqslant \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \sum_{i=1}^{I} \|\boldsymbol{v}_i\|_{\boldsymbol{L}^2(\Omega)} \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)},$$

where the constant C only depends on $\tilde{\Omega}_i$. The uniqueness of the decomposition follows from the stability estimate. □

Theorem 4.1. *Let* (H1)–(H2) *be satisfied and let* $\boldsymbol{H}_s \in \boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))$ *satisfy* (22) *and* $\boldsymbol{H}_s(\cdot,0) = 0$. *Then* (30) *has a unique solution* $\tilde{\boldsymbol{H}} \in \boldsymbol{H}^1(0,T;\tilde{\boldsymbol{X}})$ *and*

$$\|\tilde{\boldsymbol{H}}\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))} + \|\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))} \leqslant C\|\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}.$$

Proof. The proof is similar to that of Lemma 3.1. We omit the details here. □

5. Convergence of the approximate solution

This section is to show that the solution of (12) converges to the solution of (30) as the thickness of the coating film tends to zero. For simplicity, we assume that $d = \mathrm{dist}(\Omega_i;\Omega_{i+1})$ is constant for all $1 \leqslant i < I$ and denote the solution of (12) by $\boldsymbol{H}^{(d)}$. We first consider the convergence for the nonlinear eddy current problem, and then deduce an explicit error estimate for the linear eddy current problem.

5.1. Convergence for the nonlinear eddy current problem

We begin with a useful lemma.

Lemma 5.1. *There exists an* $\boldsymbol{H}^{(0)} \in \boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})$ *such that*

$$\lim_{d\to 0} \boldsymbol{H}^{(d)} = \boldsymbol{H}^{(0)} \quad \textit{strongly in } \boldsymbol{L}^2\big(0,T;\boldsymbol{L}^2(\Omega)\big) \textit{ and weakly in } \boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}}).$$

Proof. Since $\Omega_i \subset \tilde{\Omega}_i$, we have $\boldsymbol{H}_0(\mathbf{curl},\Omega_i) \subset \boldsymbol{H}_0(\mathbf{curl},\tilde{\Omega}_i)$ for all $1 \leqslant i \leqslant I$ and thus $\boldsymbol{X} \subset \tilde{\boldsymbol{X}}$. We infer from (16) that $\{\boldsymbol{H}^{(d)}\}_{d>0}$ constructs a bounded sequence in $\boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}})$. Then there exists a subsequence still denoted by $\{\boldsymbol{H}^{(d)}\}_{d>0}$ such that

$$\lim_{d\to 0} \boldsymbol{H}^{(d)} = \boldsymbol{H}^{(0)} \quad \text{weakly in } \boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}}).$$

It follows from Lemma 4.1 that $\boldsymbol{H}^{(d)}$ and $\boldsymbol{H}^{(0)}$ can be decomposed uniquely into

$$\boldsymbol{H}^{(d)} = \sum_{i=1}^{I} \boldsymbol{u}_i^{(d)} + \nabla\psi^{(d)}, \qquad \boldsymbol{H}^{(0)} = \sum_{i=1}^{I} \boldsymbol{u}_i^{(0)} + \nabla\psi^{(0)},$$

where $\boldsymbol{u}_i^{(d)}, \boldsymbol{u}_i^{(0)} \in \boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}}_i)$ and $\psi^{(d)}, \psi^{(0)} \in H^1(\Omega)/\mathbb{R}$. The uniqueness of the decompositions indicates that

$$\lim_{d\to 0} \boldsymbol{u}_i^{(d)} = \boldsymbol{u}_i^{(0)} \quad \text{weakly in } \boldsymbol{L}^2(0,T;\tilde{\boldsymbol{X}}_i).$$

Using Theorem 3.1 and the stability of the decompositions, we have

$$\sum_{i=1}^{I}\left\|\frac{\partial \boldsymbol{u}_i^{(d)}}{\partial t}\right\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}+\sum_{i=1}^{I}\|\boldsymbol{u}_i^{(d)}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))}\leqslant C\|\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}.$$

The embedding of $\boldsymbol{H}_0(\mathbf{curl},\tilde{\Omega}_i)\cap\boldsymbol{H}(\mathrm{div},\tilde{\Omega}_i)\subset\boldsymbol{H}^1(\tilde{\Omega}_i)$ [5] shows that

$$\sum_{i=1}^{I}\|\boldsymbol{u}_i^{(d)}\|_{\boldsymbol{W}^{1,2,2}(0,T;\boldsymbol{H}^1(\tilde{\Omega}_i),\boldsymbol{L}^2(\tilde{\Omega}_i))}\leqslant C\|\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}.$$

By the compact embedding $\boldsymbol{W}^{1,2,2}(0,T;\boldsymbol{H}^1(\tilde{\Omega}_i),\boldsymbol{L}^2(\tilde{\Omega}_i))\Subset\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\tilde{\Omega}_i))$, there exists a subsequence still denoted by $\{\boldsymbol{u}_i^{(d)}\}_{d>0}$ such that

$$\lim_{d\to 0}\boldsymbol{u}_i^{(d)}=\boldsymbol{u}_i^{(0)}\quad\begin{array}{l}\text{strongly in } \boldsymbol{L}^2\big(0,T;\boldsymbol{L}^2(\tilde{\Omega}_i)\big),\\ \text{weakly in } \boldsymbol{W}^{1,2,2}\big(0,T;\boldsymbol{H}^1(\tilde{\Omega}_i),\boldsymbol{L}^2(\tilde{\Omega}_i)\big).\end{array}$$

Define $\boldsymbol{B}^{(d)}:=\boldsymbol{B}(\boldsymbol{H}^{(d)}+\boldsymbol{H}_s)$. Taking test functions from $\nabla H^1(\Omega)$ in (12) shows that

$$\big(\boldsymbol{B}^{(d)},\nabla\varphi\big)=0\quad\text{for all }\varphi\in H^1(\Omega).$$

We have from assumption (H2) that

$$\begin{aligned}&\mu_{\min}\lim_{d\to 0}\|\boldsymbol{H}^{(d)}-\boldsymbol{H}^{(0)}\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}\\&\leqslant\lim_{d\to 0}\int_0^T\big(\boldsymbol{B}^{(d)}-\boldsymbol{B}^{(0)},\boldsymbol{H}^{(d)}-\boldsymbol{H}^{(0)}\big)\\&=\lim_{d\to 0}\sum_{i=1}^{I}\int_0^T\int_{\Omega_i}\boldsymbol{B}^{(d)}\cdot\big(\boldsymbol{u}_i^{(d)}-\boldsymbol{u}_i^{(0)}\big)-\lim_{d\to 0}\int_0^T\big(\boldsymbol{B}^{(0)},\boldsymbol{H}^{(d)}-\boldsymbol{H}^{(0)}\big)=0,\end{aligned}$$

where we have used the strong convergence of $\boldsymbol{u}_i^{(d)}$ and the weak convergence of $\boldsymbol{H}^{(d)}$ in the last equality. This completes the proof. □

Theorem 5.1. *Let* (H1)–(H2) *be satisfied and let* $\boldsymbol{H}^{(d)}$, $\tilde{\boldsymbol{H}}$ *be the solutions of* (12) *and* (30) *respectively. Then*

$$\lim_{d\to 0}\|\boldsymbol{H}^{(d)}-\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}=0.$$

Proof. Denote $\boldsymbol{B}^{(d)}:=\boldsymbol{B}(\boldsymbol{H}^{(d)}+\boldsymbol{H}_s)$ and $\tilde{\boldsymbol{B}}^{(d)}:=\tilde{\boldsymbol{B}}(\boldsymbol{H}^{(d)}+\boldsymbol{H}_s)$ for any $d\geqslant 0$, where $\tilde{\boldsymbol{B}}$ is defined in (29). Following from (H2) and Theorem 3.1, we have

$$\|\tilde{\boldsymbol{B}}^{(d)}\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}\leqslant C\|\boldsymbol{H}^{(d)}+\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}\leqslant C\|\boldsymbol{H}_s\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}.\tag{31}$$

Since $\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))$ is self-reflective, there exist a $\boldsymbol{B}_0 \in \boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))$ and a subsequence of $\{\tilde{\boldsymbol{B}}^{(d)}\}_{d>0}$ such that

$$\lim_{d\to 0} \tilde{\boldsymbol{B}}^{(d)} = \boldsymbol{B}_0 \quad \text{weakly in } \boldsymbol{H}^1\big(0,T;\boldsymbol{L}^2(\Omega)\big). \tag{32}$$

Using assumption (H2) and Lemma 5.1, we obtain that

$$\lim_{d\to 0} \|\tilde{\boldsymbol{B}}^{(d)} - \tilde{\boldsymbol{B}}^{(0)}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \leqslant \mu_{\max} \lim_{d\to 0} \|\boldsymbol{H}^{(d)} - \boldsymbol{H}^{(0)}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} = 0,$$

where $\boldsymbol{H}^{(0)}$ is the limit of $\boldsymbol{H}^{(d)}$ in Lemma 5.1. Thus we conclude that

$$\boldsymbol{B}_0 = \tilde{\boldsymbol{B}}^{(0)} = \tilde{\boldsymbol{B}}\big(\boldsymbol{H}^{(0)} + \boldsymbol{H}_s\big).$$

Noting the measure $(\tilde{\Omega}_i \backslash \overline{\Omega}_i) \to 0$ as $d \to 0$, we have from (31) that

$$\lim_{d\to 0} \int_0^T \int_{\tilde{\Omega}_i \backslash \overline{\Omega}_i} \frac{\partial}{\partial t}\big(\boldsymbol{B}^{(d)} - \tilde{\boldsymbol{B}}^{(d)}\big) \cdot \boldsymbol{v} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2\big(0,T;\boldsymbol{L}^2(\Omega)\big),$$

which implies that

$$\lim_{d\to 0} \int_0^T \left(\frac{\partial}{\partial t}\boldsymbol{B}^{(d)}, \boldsymbol{v}\right) = \lim_{d\to 0} \int_0^T \left(\frac{\partial}{\partial t}\tilde{\boldsymbol{B}}^{(d)}, \boldsymbol{v}\right) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2\big(0,T;\boldsymbol{L}^2(\Omega)\big). \tag{33}$$

From (12) and $\operatorname{supp}(\mathbf{curl}\,\boldsymbol{H}^{(d)}) = \operatorname{supp}(\sigma_1)$, we deduce that $\boldsymbol{H}^{(d)}$ satisfies

$$\int_0^T \left(\frac{\partial}{\partial t}\boldsymbol{B}^{(d)}, \boldsymbol{v}\right) + \int_0^T \big(\tilde{\sigma}_1\,\mathbf{curl}\,\boldsymbol{H}^{(d)}, \mathbf{curl}\,\boldsymbol{v}\big) = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{L}^2\big(0,T;\boldsymbol{L}^2(\Omega)\big).$$

Taking the limit of both sides as $d \to 0$ and using (32) and (33), we obtain that $\boldsymbol{H}^{(0)}$ satisfies (30).

Furthermore, the initial condition that $\boldsymbol{H}^{(0)}(\cdot,0) = 0$ can be proved by similar arguments as in the proof of Theorem 3.1. We conclude that $\tilde{\boldsymbol{H}} := \boldsymbol{H}^{(0)}$ is the unique solution of (30). □

5.2. Error estimate for the linear eddy current problem

In this section, we are concerned with the linear eddy current problem for laminated conductors, and intend to estimate the approximation error with respect to the thickness of the coating film. For the linear model problem, it is assumed that $\boldsymbol{B}(\boldsymbol{H}) = \mu_0 \boldsymbol{H}$ in Ω where μ_0 is the magnetic permeability of the vacuum.

Similar to (12), the weak formulation for isolated conductors may be formulated as follows: Find $\boldsymbol{H} \in \boldsymbol{X}$ such that

$$\int_\Omega \mu_0 \frac{\partial \boldsymbol{H}}{\partial t} \cdot \boldsymbol{v} + \int_\Omega \sigma_1\,\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\boldsymbol{v} = -\int_\Omega \mu_0 \frac{\partial \boldsymbol{H}_s}{\partial t} \cdot \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}. \tag{34}$$

Comparing with the weak formulation for the approximate nonlinear problem (30), we have the approximate linear problem for extended (or adjacent) conductors: Find $\tilde{\boldsymbol{H}} \in \tilde{\boldsymbol{X}}$ such that

$$\int_{\Omega} \mu_0 \frac{\partial \tilde{\boldsymbol{H}}}{\partial t} \cdot \boldsymbol{v} + \int_{\Omega} \tilde{\sigma}_1 \,\mathbf{curl}\, \tilde{\boldsymbol{H}} \cdot \mathbf{curl}\, \boldsymbol{v} = -\int_{\Omega} \mu_0 \frac{\partial \boldsymbol{H}_s}{\partial t} \cdot \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in \tilde{\boldsymbol{X}}. \tag{35}$$

Lemma 5.2. *Let $\boldsymbol{H} \in \boldsymbol{H}^1(0,T;\boldsymbol{X})$ and $\tilde{\boldsymbol{H}} \in \boldsymbol{H}^1(0,T;\tilde{\boldsymbol{X}})$ be the solutions of (34) and (35) respectively. Then there exists a constant $C > 0$ only depending on T, $\tilde{\Omega}_c$, $\sigma_{\max}$, $\sigma_{\min}$ such that*

$$\begin{aligned}
&\|\tilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \big\|\mathbf{curl}(\tilde{\boldsymbol{H}} - \boldsymbol{H})\big\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\quad \leqslant C \inf_{\substack{\boldsymbol{v} \in \boldsymbol{H}^1(0,T;\boldsymbol{X}) \\ \boldsymbol{v}(0)=0}} \big\{\|\tilde{\boldsymbol{H}} - \boldsymbol{v}\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))} + \big\|\mathbf{curl}(\tilde{\boldsymbol{H}} - \boldsymbol{v})\big\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}\big\}.
\end{aligned}$$

Proof. It is clear that $\boldsymbol{H}$ is the Galerkin approximation to $\tilde{\boldsymbol{H}}$ in the subspace $\boldsymbol{X} \subset \tilde{\boldsymbol{X}}$. Denote the error function by $\boldsymbol{h} := \tilde{\boldsymbol{H}} - \boldsymbol{H}$. Subtracting (34) from (35) yields

$$\int_{\Omega} \left(\frac{\partial \boldsymbol{h}}{\partial t} \cdot \boldsymbol{v} + \tilde{\sigma}_1 \,\mathbf{curl}\, \boldsymbol{h} \cdot \mathbf{curl}\, \boldsymbol{v} \right) = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}.$$

Taking $\boldsymbol{v} \in \boldsymbol{H}^1(0,T;\boldsymbol{X})$ with $\boldsymbol{v}(\cdot,0) = 0$ and integrating the above equality over $(0,t)$, we have

$$\begin{aligned}
&\big\|\boldsymbol{h}(t)\big\|^2_{\boldsymbol{L}^2(\Omega)} + \int_0^t \int_{\Omega} \tilde{\sigma}_1 |\mathbf{curl}\, \boldsymbol{h}|^2 \\
&\quad = \int_0^t \int_{\Omega} \left\{ \frac{\partial \boldsymbol{h}}{\partial t} \cdot (\boldsymbol{h} - \boldsymbol{v}) + \tilde{\sigma}_1 \,\mathbf{curl}\, \boldsymbol{h} \cdot \mathbf{curl}(\boldsymbol{h} - \boldsymbol{v}) \right\} \\
&\quad = \int_{\Omega} \boldsymbol{h}(t)\big\{\boldsymbol{h}(t) - \boldsymbol{v}(t)\big\} + \int_0^t \int_{\Omega} \left\{ \tilde{\sigma}_1 \,\mathbf{curl}\, \boldsymbol{h} \cdot \mathbf{curl}(\boldsymbol{h} - \boldsymbol{v}) - \boldsymbol{h} \cdot \frac{\partial(\boldsymbol{h} - \boldsymbol{v})}{\partial t} \right\} \\
&\quad \leqslant \frac{1}{2} \left\{ \big\|\boldsymbol{h}(t)\big\|^2_{\boldsymbol{L}^2(\Omega)} + \int_0^t \int_{\Omega} \tilde{\sigma}_1 |\mathbf{curl}\, \boldsymbol{h}|^2 \right\} + \frac{1}{2} \int_0^t \|\boldsymbol{h}\|^2_{\boldsymbol{L}^2(\Omega)} \\
&\qquad + 2\big\|\boldsymbol{h}(t) - \boldsymbol{v}(t)\big\|^2_{\boldsymbol{L}^2(\Omega)} + 2 \int_0^t \int_{\Omega} \left\{ \tilde{\sigma}_1 \big|\mathbf{curl}(\boldsymbol{h} - \boldsymbol{v})\big|^2 + \left| \frac{\partial(\boldsymbol{h} - \boldsymbol{v})}{\partial t} \right|^2 \right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\big\|\boldsymbol{h}(t)\big\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{h}\|^2_{\boldsymbol{L}^2(0,t;\boldsymbol{L}^2(\tilde{\Omega}_c))} &\leqslant \frac{1}{2} \int_0^t \|\boldsymbol{h}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{h} - \boldsymbol{v}\|^2_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))} \\
&\quad + C\big\|\mathbf{curl}(\boldsymbol{h} - \boldsymbol{v})\big\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} + C\|\boldsymbol{h} - \boldsymbol{v}\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}.
\end{aligned}$$

Using the initial condition for $\boldsymbol{h}-\boldsymbol{v}$, we have

$$\left|\boldsymbol{h}(s)-\boldsymbol{v}(s)\right|^2=\left|\int_0^s \frac{\partial}{\partial t}(\boldsymbol{h}-\boldsymbol{v})\,\mathrm{d}t\right|^2 \leqslant T\int_0^T\left|\frac{\partial}{\partial t}(\boldsymbol{h}-\boldsymbol{v})\right|^2\mathrm{d}t \quad \text{for all } 0\leqslant s\leqslant T,$$

which implies that

$$\begin{aligned}\left\|\boldsymbol{h}(t)\right\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mathbf{curl}\,\boldsymbol{h}\|^2_{\boldsymbol{L}^2(0,t;\boldsymbol{L}^2(\tilde{\Omega}_c))} &\leqslant \frac{1}{2}\int_0^t \|\boldsymbol{h}\|^2_{\boldsymbol{L}^2(\Omega)}+C\|\boldsymbol{h}-\boldsymbol{v}\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}\\ &\quad +C\left\|\mathbf{curl}(\boldsymbol{h}-\boldsymbol{v})\right\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}.\end{aligned}$$

The proof is completed by an application of Gronwall's inequality and the arbitrariness of t. □

Next we derive the convergence rate by finding a proper candidate of $\boldsymbol{v}$ in the infimum in Lemma 5.2. First we write

$$\tilde{\boldsymbol{H}}=\nabla\psi+\sum_{i=1}^I \boldsymbol{u}_i, \quad \psi\in H^1(\Omega),\ \boldsymbol{u}_i\in \boldsymbol{H}_0(\mathbf{curl},\tilde{\Omega}_i).$$

Let $\phi_i\in H_0^1(\tilde{\Omega}_i)$ be the unique solution of the elliptic problem

$$\int_{\tilde{\Omega}_i}\nabla\phi_i\cdot\nabla v=\int_{\tilde{\Omega}_i}\boldsymbol{u}_i\cdot\nabla v \quad \text{for all } v\in H_0^1(\tilde{\Omega}_i).$$

Define $\boldsymbol{w}_i:=\boldsymbol{u}_i-\nabla\phi_i$. Clearly we have

$$\operatorname{div}\boldsymbol{w}_i=0 \quad \text{in } \tilde{\Omega}_i \quad \text{and} \quad \boldsymbol{w}_i\in\boldsymbol{H}_0(\mathbf{curl},\tilde{\Omega}_i)\cap\boldsymbol{H}(\operatorname{div},\tilde{\Omega}_i).$$

By the imbedding theorem in [5], we know that $\boldsymbol{w}_i\in\boldsymbol{H}^1(\tilde{\Omega}_i)$ and

$$\|\boldsymbol{w}_i\|_{\boldsymbol{H}^1(\tilde{\Omega}_i)}\leqslant C\big(\|\mathbf{curl}\,\boldsymbol{w}_i\|_{\boldsymbol{L}^2(\tilde{\Omega}_i)}+\|\operatorname{div}\boldsymbol{w}_i\|_{\boldsymbol{L}^2(\tilde{\Omega}_i)}\big)=C\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\tilde{\Omega}_i)}, \tag{36}$$

where the constant C only depends on $\tilde{\Omega}_i$. We extend ϕ_i, $\boldsymbol{w}_i$ by zeros to the exterior of $\tilde{\Omega}_i$ and define $\phi:=\sum_{i=1}^I\phi_i\in H^1(\Omega)$. Then we have

$$\tilde{\boldsymbol{H}}=\nabla(\psi+\phi)+\sum_{i=1}^I\boldsymbol{w}_i. \tag{37}$$

Lemma 5.3. *Let $\tilde{\boldsymbol{H}}\in\boldsymbol{H}^1(0,T;\tilde{\boldsymbol{X}})$ be the solution of (35) and assume $\mathbf{curl}\,\tilde{\boldsymbol{H}}\in\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))$. There exists a constant C independent of d such that*

$$\inf_{\substack{\boldsymbol{v}\in \boldsymbol{H}^1(0,T;\boldsymbol{X})\\ \boldsymbol{v}(0)=0}} \|\tilde{\boldsymbol{H}}-\boldsymbol{v}\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))} \leqslant Cd^{1/3}\|\mathbf{curl}\,\tilde{\boldsymbol{H}}\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}, \tag{38}$$

$$\inf_{\substack{\boldsymbol{v}\in \boldsymbol{H}^1(0,T;\boldsymbol{X})\\ \boldsymbol{v}(0)=0}} \left\|\mathbf{curl}(\tilde{\boldsymbol{H}}-\boldsymbol{v})\right\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \leqslant Cd^{1/3}\|\mathbf{curl}\,\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))}. \tag{39}$$

Proof. We define a coordinate stretching $F_i\colon \Omega_i \mapsto \tilde{\Omega}_i$ by

$$\boldsymbol{x} = F_i(\hat{\boldsymbol{x}}) := \mathbb{B}_i\hat{\boldsymbol{x}} - \boldsymbol{b}_i \quad \text{for all } \hat{\boldsymbol{x}} \in \Omega_i,$$

where

$$\mathbb{B}_i = \operatorname{diag}\left(1, 1, \frac{Z_i - Z_{i-1}}{Z_i - Z_{i-1} - d}\right), \qquad \boldsymbol{b}_i = \left(0, 0, \frac{Z_i d}{Z_i - Z_{i-1} - d}\right)^T. \tag{40}$$

Let $\boldsymbol{w}_i \in \boldsymbol{H}_0(\mathbf{curl}, \tilde{\Omega}_i)$ be the splitting component of $\tilde{\boldsymbol{H}}$ given in (37) and define

$$\hat{\boldsymbol{w}}_i := \mathbb{B}_i(\boldsymbol{w}_i \circ F_i).$$

Direct calculations show that

$$\widehat{\mathbf{curl}}\,\hat{\boldsymbol{w}}_i = \frac{Z_i - Z_{i-1}}{Z_i - Z_{i-1} - d}\mathbb{B}_i^{-1}(\mathbf{curl}\,\boldsymbol{w}_i \circ F_i). \tag{41}$$

Since the unit outer normals of $\partial\Omega_i$ and $\partial\tilde{\Omega}_i$ have the range

$$\left\{\pm(1,0,0)^T, \pm(0,1,0)^T, \pm(0,0,1)^T\right\},$$

we deduce that

$$(\hat{\boldsymbol{w}}_i \times \hat{\boldsymbol{n}})|_{\partial\Omega_i} = \left\{\mathbb{B}_i(\boldsymbol{w}_i \circ F_i) \times \hat{\boldsymbol{n}}\right\}\big|_{\partial\Omega_i} = \{\mathbb{B}_i\boldsymbol{w}_i \times \boldsymbol{n}\}|_{\partial\tilde{\Omega}_i} = \mathbb{C}_{\boldsymbol{n}}(\boldsymbol{w}_i \times \boldsymbol{n})|_{\partial\tilde{\Omega}_i} = 0,$$

where $\mathbb{C}_{\boldsymbol{n}}$ is a diagonal matrix and each diagonal entry of $\mathbb{C}_{\boldsymbol{n}}$ is either 1 or $(Z_i d)/(Z_i - Z_{i-1} - d)$ according to the variation of $\boldsymbol{n}$. Thus $\hat{\boldsymbol{w}}_i \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_i)$.

From (37), the first inequality in Lemma 5.3 is estimated as follows

$$\begin{aligned}
&\inf_{\substack{\boldsymbol{v}\in \boldsymbol{H}^1(0,T;\boldsymbol{X})\\ \boldsymbol{v}(0)=0}} \|\tilde{\boldsymbol{H}}-\boldsymbol{v}\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}\\
&\leqslant \sum_{i=1}^{I} \|\boldsymbol{w}_i - \hat{\boldsymbol{w}}_i\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\tilde{\Omega}_i))}\\
&\leqslant \sum_{i=1}^{I} \left\{\|\boldsymbol{w}_i - \mathbb{B}_i\boldsymbol{w}_i\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\tilde{\Omega}_i))} + \left\|\mathbb{B}_i(\boldsymbol{w}_i - \boldsymbol{w}_i \circ F_i)\right\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\tilde{\Omega}_i))}\right\}.
\end{aligned} \tag{42}$$

We have from (40) that

$$\|\boldsymbol{w}_i - \mathbb{B}_i\boldsymbol{w}_i\|_{\boldsymbol{L}^2(\tilde{\Omega}_i)} \leqslant \frac{d}{Z_i - Z_{i-1} - d}\|\boldsymbol{w}_i\|_{\boldsymbol{L}^2(\tilde{\Omega}_i)}.$$

Furthermore, there is a constant C independent of d such that

$$\left|\boldsymbol{x} - F_i(\boldsymbol{x})\right| \leqslant \left|(1-\mathbb{B}_i)\boldsymbol{x}\right| + |\boldsymbol{b}_i| \leqslant Cd \quad \text{for all } \boldsymbol{x} \in \Omega_i.$$

Since $\boldsymbol{w}_i \circ F_i = 0$ in $\tilde{\Omega}_i \backslash \bar{\Omega}_i$, we have

$$\|\boldsymbol{w}_i - \boldsymbol{w}_i \circ F_i\|^2_{\boldsymbol{L}^2(\tilde{\Omega}_i)} = \|\boldsymbol{w}_i - \boldsymbol{w}_i \circ F_i\|^2_{\boldsymbol{L}^2(\Omega_i)} + \|\boldsymbol{w}_i\|^2_{\boldsymbol{L}^2(\tilde{\Omega}_i \backslash \bar{\Omega}_i)}.$$

The above two terms are estimated as follows

$$\|\boldsymbol{w}_i\|^2_{\boldsymbol{L}^2(\tilde{\Omega}_i \backslash \bar{\Omega}_i)} \leqslant Cd^{2/3}\|\boldsymbol{w}_i\|^2_{\boldsymbol{L}^6(\tilde{\Omega}_i \backslash \bar{\Omega}_i)} \leqslant Cd^{2/3}\|\boldsymbol{w}_i\|^2_{\boldsymbol{L}^6(\tilde{\Omega}_i)} \leqslant Cd^{2/3}\|\boldsymbol{w}_i\|^2_{\boldsymbol{H}^1(\tilde{\Omega}_i)},$$

$$\begin{aligned}
\|\boldsymbol{w}_i - \boldsymbol{w}_i \circ F_i\|^2_{\boldsymbol{L}^2(\Omega_i)} &= \int_{\Omega_i} \left| \int_0^1 \nabla \boldsymbol{w}_i\big(t\boldsymbol{x} + (1-t)F_i(\boldsymbol{x})\big)\,\mathrm{d}t \cdot \big[\boldsymbol{x} - F_i(\boldsymbol{x})\big] \right|^2 \mathrm{d}\boldsymbol{x} \\
&\leqslant Cd^2 \int_{\Omega_i} \int_0^1 \left|\nabla \boldsymbol{w}_i\big(t\boldsymbol{x} + (1-t)F_i(\boldsymbol{x})\big)\right|^2 \mathrm{d}t\,\mathrm{d}\boldsymbol{x} \\
&\leqslant Cd^2 \|\boldsymbol{w}_i\|^2_{\boldsymbol{H}^1(\tilde{\Omega}_i)}.
\end{aligned}$$

We conclude from (37) that

$$\sum_{i=1}^{I} \|\boldsymbol{w}_i - \hat{\boldsymbol{w}}_i\|^2_{\boldsymbol{L}^2(\tilde{\Omega}_i)} \leqslant Cd^{2/3} \sum_{i=1}^{I} \|\boldsymbol{w}_i\|^2_{\boldsymbol{H}^1(\tilde{\Omega}_i)} \leqslant Cd^{2/3} \|\mathbf{curl}\,\tilde{\boldsymbol{H}}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{43}$$

Plugging (43) into (42) yields (38).

Furthermore, observe that

$$\mathbf{curl}\,\boldsymbol{w}_i \cdot \boldsymbol{n} = \mathrm{Curl}_s\big(\boldsymbol{n} \times (\boldsymbol{w}_i \times \boldsymbol{n})\big) = 0 \quad \text{on } \partial\tilde{\Omega}_i,$$

where Curl_s is the surface curl operator on $\partial\tilde{\Omega}_i$. Thus we get

$$\mathbf{curl}\,\tilde{\boldsymbol{H}} = \mathbf{curl}\,\boldsymbol{w}_i \in \boldsymbol{H}_0(\mathrm{div}, \tilde{\Omega}_i) \cap \boldsymbol{H}(\mathbf{curl}, \tilde{\Omega}_i) \subset \boldsymbol{H}^1(\tilde{\Omega}_i).$$

It can be verified that

$$\inf_{\substack{\boldsymbol{v} \in \boldsymbol{H}^1(0,T;\boldsymbol{X}) \\ \boldsymbol{v}(0)=0}} \left\|\mathbf{curl}(\tilde{\boldsymbol{H}} - \boldsymbol{v})\right\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))} \leqslant \sum_{i=1}^{I} \left\|\mathbf{curl}(\boldsymbol{w}_i - \hat{\boldsymbol{w}}_i)\right\|^2_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\tilde{\Omega}_i))}.$$

Then (39) can be proved by (41) and similar arguments. We do not elaborate on the details here. This completes the proof. □

A direct consequence of Lemmas 5.2 and 5.3 gives the following result on the convergence rate for the approximate solution of the linear eddy current problem.

Theorem 5.2. *Let $\boldsymbol{H} \in \boldsymbol{H}^1(0,T;\boldsymbol{X})$ and $\tilde{\boldsymbol{H}} \in \boldsymbol{H}^1(0,T;\tilde{\boldsymbol{X}})$ be the solutions of (34) and (35) respectively and assume $\mathbf{curl}\,\tilde{\boldsymbol{H}} \in \boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))$. Then there exists a constant $C > 0$ only depending on T, $\tilde{\Omega}_c$, $\sigma_{\max}$, $\sigma_{\min}$ such that*

$$\begin{aligned}&\|\tilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \left\|\mathbf{curl}(\tilde{\boldsymbol{H}} - \boldsymbol{H})\right\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}\\ &\quad\leqslant Cd^{1/3}\left\{\|\mathbf{curl}\,\tilde{\boldsymbol{H}}\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))} + \|\mathbf{curl}\,\tilde{\boldsymbol{H}}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))}\right\}.\end{aligned}$$

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Appendix A

The purpose of this appendix is to establish the well-posedness of the semi-discrete problem (13). First we need the following theorem on strongly monotone operators.

Theorem A.1. *(See [30, Theorem 25.B].) Let X be a real Banach space and let $A\colon X \to X'$ be an operator satisfying*

- *strong monotonicity*:

$$\langle Au - Av, u - v\rangle \geqslant c\|u - v\|_X^2 \quad \textit{for all } u, v \in X,$$

- *Lipschitz continuity*:

$$\|Au - Av\|_{X'} \leqslant L\|u - v\|_X \quad \textit{for all } u, v \in X,$$

where the constants $c > 0$, $L > 0$ only depends on A and X. Then for any $b \in X'$, the operator equation $Au = b$ has a unique solution $u \in X$.

Here is the proof of Lemma 3.1.

Proof of Lemma 3.1. First we write (13) as: Find $\boldsymbol{H}_n \in \boldsymbol{X}$ such that

$$(\boldsymbol{B}_n, \boldsymbol{v}) + \tau(\sigma_1\,\mathbf{curl}\,\boldsymbol{H}_n, \mathbf{curl}\,\boldsymbol{v}) = (\boldsymbol{B}_{n-1}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}, \tag{A.1}$$

where $\boldsymbol{B}_n := \boldsymbol{B}(\boldsymbol{H}_n + \boldsymbol{H}_s(t_n))$.

For any $\boldsymbol{w} \in \boldsymbol{X}$, let $\mathcal{L}_n\boldsymbol{w} \in \boldsymbol{X}$ be the unique solution of the variational problem

$$(\mathcal{L}_n\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{H}(\mathbf{curl},\Omega)} = \big(\boldsymbol{B}_n(\boldsymbol{w}), \boldsymbol{v}\big) + \tau(\sigma_1\,\mathbf{curl}\,\boldsymbol{w}, \mathbf{curl}\,\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}, \tag{A.2}$$

where $\boldsymbol{B}_n(\boldsymbol{w}) := \boldsymbol{B}(\boldsymbol{w} + \boldsymbol{H}_s(t_n))$. Let $\boldsymbol{f}_n \in \boldsymbol{X}$ be the unique solution of the variational problem

$$(\boldsymbol{f}_n, \boldsymbol{v})_{\boldsymbol{H}(\mathbf{curl},\Omega)} = (\boldsymbol{B}_{n-1}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}.$$

Clearly (A.1) is equivalent to the operator equation

$$\mathcal{L}_n\boldsymbol{H}_n = \boldsymbol{f}_n \quad \text{in } \boldsymbol{X}. \tag{A.3}$$

From (H2) we infer that the operator $\mathcal{L}_n\colon \boldsymbol{X} \to \boldsymbol{X}$ is Lipschitz continuous. Moreover, the strict monotonicity of $\mathcal{L}_n$ comes directly from (H1)–(H2): for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{X}$,

$$\begin{aligned}(\mathcal{L}_n\boldsymbol{u} - \mathcal{L}_n\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_{\boldsymbol{H}(\mathbf{curl},\Omega)} &= \big(\boldsymbol{B}_n(\boldsymbol{u}) - \boldsymbol{B}_n(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}\big) + \tau\big(\sigma_1\,\mathbf{curl}(\boldsymbol{u} - \boldsymbol{v}), \mathbf{curl}(\boldsymbol{u} - \boldsymbol{v})\big)\\ &\geqslant \min(\mu_{\min}, \tau\sigma_{\min})\|\boldsymbol{u} - \boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)}.\end{aligned}$$

Then Theorem A.1 shows that (A.3) or (A.1) has a unique solution for each $n \geqslant 1$.

Setting $\boldsymbol{v} = \boldsymbol{H}_n - \boldsymbol{H}_{n-1}$ in (A.1) shows that

$$\tau^{-1}(\boldsymbol{B}_n - \boldsymbol{B}_{n-1}, \boldsymbol{H}_n - \boldsymbol{H}_{n-1}) + (\sigma_1\,\mathbf{curl}\,\boldsymbol{H}_n, \mathbf{curl}\,\boldsymbol{H}_n - \mathbf{curl}\,\boldsymbol{H}_{n-1}) = 0. \tag{A.4}$$

Using the initial value $\boldsymbol{H}_0 = 0$ and the inequality

$$2\big(\sigma_1\,\mathbf{curl}\,\boldsymbol{H}_n, \mathbf{curl}(\boldsymbol{H}_n - \boldsymbol{H}_{n-1})\big) \geqslant \big\|\sigma_1^{\frac{1}{2}}\,\mathbf{curl}\,\boldsymbol{H}_n\big\|^2_{L^2(\Omega)} - \big\|\sigma_1^{\frac{1}{2}}\,\mathbf{curl}\,\boldsymbol{H}_{n-1}\big\|^2_{L^2(\Omega)},$$

we have

$$\sum_{n=1}^{m}(\sigma_1\,\mathbf{curl}\,\boldsymbol{H}_n, \mathbf{curl}\,\boldsymbol{H}_n - \mathbf{curl}\,\boldsymbol{H}_{n-1}) \geqslant \frac{1}{2\sigma_{\max}}\|\mathbf{curl}\,\boldsymbol{H}_m\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{A.5}$$

Denote the approximate magnetic field by $\hat{\boldsymbol{H}}_n := \boldsymbol{H}_n + \boldsymbol{H}_s(t_n)$. By (H2), we have

$$\begin{aligned}&(\boldsymbol{B}_n - \boldsymbol{B}_{n-1}, \boldsymbol{H}_n - \boldsymbol{H}_{n-1})\\ &\quad= \big(\boldsymbol{B}(\hat{\boldsymbol{H}}_n) - \boldsymbol{B}(\hat{\boldsymbol{H}}_{n-1}), \hat{\boldsymbol{H}}_n - \hat{\boldsymbol{H}}_{n-1}\big) - \big(\boldsymbol{B}(\hat{\boldsymbol{H}}_n) - \boldsymbol{B}(\hat{\boldsymbol{H}}_{n-1}), \boldsymbol{H}_s(t_n) - \boldsymbol{H}_s(t_{n-1})\big)\\ &\quad\geqslant \mu_{\min}\|\hat{\boldsymbol{H}}_n - \hat{\boldsymbol{H}}_{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} - \mu_{\max}\|\hat{\boldsymbol{H}}_n - \hat{\boldsymbol{H}}_{n-1}\|_{\boldsymbol{L}^2(\Omega)}\big\|\boldsymbol{H}_s(t_n) - \boldsymbol{H}_s(t_{n-1})\big\|_{\boldsymbol{L}^2(\Omega)}\\ &\quad\geqslant \frac{\mu_{\min}}{2}\|\hat{\boldsymbol{H}}_n - \hat{\boldsymbol{H}}_{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{2\tau\mu_{\max}^2}{\mu_{\min}}\int_{t_{n-1}}^{t_n}\left\|\frac{\partial\boldsymbol{H}_s}{\partial t}\right\|^2_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t.\end{aligned} \tag{A.6}$$

Plugging (A.5) and (A.6) into (A.4), we obtain

$$\sum_{n=1}^{m}\frac{1}{\tau}\|\hat{\boldsymbol{H}}_n - \hat{\boldsymbol{H}}_{n-1}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\|\mathbf{curl}\,\boldsymbol{H}_m\|^2_{\boldsymbol{L}^2(\Omega)}}{\mu_{\min}\sigma_{\max}} \leqslant \left(\frac{2\mu_{\max}}{\mu_{\min}}\right)^2\int_0^{t_m}\left\|\frac{\partial\boldsymbol{H}_s}{\partial t}\right\|^2_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t.$$

Then (15) follows from the above inequality and the arbitrariness of m. A direct consequence of (15) gives (16).

From (H2) we deduce that

$$\begin{aligned}|\boldsymbol{B}_n| &= 1\big|\boldsymbol{B}(\hat{\boldsymbol{H}}_n) - \boldsymbol{B}(0)\big| \leqslant \mu_{\max}|\hat{\boldsymbol{H}}_n| = \mu_{\max}|\boldsymbol{H}_n + \boldsymbol{H}_s|,\\ |\boldsymbol{B}_n - \boldsymbol{B}_{n-1}| &= \big|\boldsymbol{B}(\hat{\boldsymbol{H}}_n) - \boldsymbol{B}(\hat{\boldsymbol{H}}_{n-1})\big| \leqslant \mu_{\max}\left|\frac{\partial}{\partial t}(\boldsymbol{H}_\tau + \boldsymbol{H}_s)\right|.\end{aligned}$$

Combining (15), we obtain (17). □

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